Introduction to Robotics (CS223A) Homework #6 Solution

(Winter 2007/2008)

1. **Consider the 1-DOF system described by the equation of motion,** $4\ddot{x}+20\dot{x}+25x = f$**.**

   (a) **Find the natural frequency** ω_n **and the natural damping ratio** ξ_n **of the natural (passive) system (**$f = 0$**). What type of system is this (oscillatory, overdamped, etc.) ?**

   Using Section 7.2.2 from the course reader, we can compare this sytem with $m\ddot{x} + b\dot{x} + kx = 0$ like in Equation 7.9. Thus:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{25}{4}} = 2.5$$

$$\xi_n = \frac{b}{2\sqrt{km}} = \frac{20}{2\sqrt{25 \cdot 4}} = 1$$

   Since $\xi_n = 1$, this system is **critically damped**.

   (b) **Design a PD controller that achieves critical damping with a closed-loop stiffness** $k_{CL} = 36$**. In other words, let** $f = -k_v\dot{x} - k_p x$**, and determine the gains** k_v **and** k_p**. Assume that the desired position is** $x_d = 0$**.**

   The original system is:

$$4\ddot{x} + 20\dot{x} + 25x = f$$

   with input force f. The controller provides this input, using the formula:

$$f = -k_v\dot{x} - k_p x$$

   So, the closed loop equation is:

$$4\ddot{x} + 20\dot{x} + 25x = -k_v\dot{x} - k_p x$$

$$\Rightarrow 4\ddot{x} + (20 + k_v)\dot{x} + (25 + k_p)x = 0$$

   This closed loop system behaves just like the natural dissipative system in Section 7.2.2 of the course reader. So, we first compare to Equation 7.9:

$$4\ddot{x} + (20 + k_v)\dot{x} + (25 + k_p)x = m\ddot{x} + b\dot{x} + kx = 0$$

   The closed loop stiffness is given by k, the coefficient of the positional term, so:

$$k = 25 + k_p = k_{CL} = 36$$

   For the damping requirement, we need to first figure out how the control gains k_p and k_v affect the damping ratio ξ; we do this by applying Equation 7.12 to our system:

$$\xi = \frac{b}{2\sqrt{km}}$$

For critical damping:

$$\xi = 1$$

so the coefficient b of $\dot{x}$ must satisfy

$$b = 2\sqrt{km} = 2\sqrt{36 \cdot 4} = 24$$

Based on our closed loop equation, we have:

$$b = 20 + k_v$$

so the gains that we need are

$$k_p = 11, \;\; k_v = 4$$

and the PD controller is

$$f = -4\dot{x} - 11x$$

(c) **Assume that the friction model changes from linear ($20\dot{x}$) to Coulomb friction, $30sign(\dot{x})$. Design a control system which uses a non-linear model-based portion with trajectory following to critically damp the system at all times and maintain a closed-loop stiffness of $k_{CL} = 36$. In other words, let $f = \alpha f' + \beta$ and $f' = \ddot{x}_d - k'_v(\dot{x} - \dot{x}_d) - k'_p(x - x_d)$. Then, find f,α,β,f',k'_p and k'_v. Note that f is an m-mass control, and f' is a unit-mass control. Use the definition of error, $e = x - x_d$.**

The differential equation for the system is now

$$4\ddot{x} + 30sign(\dot{x}) + 25x = f$$

In order to linearize it, we apply a force f of the form

$$f = \alpha f' + \beta$$

where

$$\alpha = 4, \;\; \beta = 30sign(\dot{x}) + 25x$$

For purposes of control, this makes the system look like the unit-mass system:

$$\ddot{x} = f'$$

to which we apply the control

$$f' = \ddot{x}_d - k'_v(\dot{x} - \dot{x}_d) - k'_p(x - x_d)$$

Substituting into our unit-mass system yields the equation

$$\ddot{e} + k'_v\dot{e} + k'_p e = 0$$

where e is the position error, $e = x - x_d$.

Now, we want to choose our gains k'_p and k'_v so that we achieve critical damping and the desired closed-loop stiffness.

To look at the closed loop stiffness, we need to consider the controlled system before factoring out the mass:

$$4\ddot{x} + 30sign(\dot{x}) + 25x = \alpha f' + \beta$$

$$\Rightarrow 4\ddot{x} = \alpha f'$$

$$\Rightarrow 4\ddot{x} + \alpha f' = 0$$

$$\Rightarrow 4(\ddot{x} - \ddot{x}_d) + 4k_v'(\dot{x} - \dot{x}_d) + 4k_p'(x - x_d) = 0$$

$$\Rightarrow 4\ddot{e} + 4k_v'\dot{e} + 4k_p'e = 0$$

The coefficient of the e term is the closed loop stiffness, so: $k_p' = 9$.
In order to have critical damping, we need to have $\xi = 1$. Using Equation 7.12 we see that the coefficient of $\dot{e}$ must be:

$$k_v' = 2\sqrt{k_p'} = 6$$

Thus, the control is

$$\begin{aligned} f &= \alpha f' + \beta \\ \alpha &= 4 \\ \beta &= 30sign(\dot{x}) + 25x \\ f' &= \ddot{x}_d - 6(\dot{x} - \dot{x}_d) - 9(x - x_d) \end{aligned}$$

(d) **Given a disturbance force $f_{dist} = 4$, what is the steady-state ($\ddot{e} = \dot{e} = 0$) error of the system in part (c)?**
We can analyze the error by observing the error in the unit-mass system. With a disturbance force added, the system's equation of motion becomes

$$4\ddot{x} + 30sign(\dot{x}) + 25x = f + f_{dist}$$

To linearize the system, we apply a force of the same form as before:

$$\begin{aligned} f + f_{dist} &= 4f' + 30sign(\dot{x}) + 25x + f_{dist} \\ &= 4\left(f' + \frac{f_{dist}}{4}\right) + 30sign(\dot{x}) + 25x \end{aligned}$$

This yields a unit-mass system as before, but now it has a disturbance force of $f_{dist}/4$, so the unit-mass system now looks like

$$\ddot{x} = f' + \frac{f_{dist}}{4}$$

With the control from before, we get a unit-mass closed-loop system of

$$\ddot{e} + 6\dot{e} + 9e = \frac{f_{dist}}{4}$$

For the steady state, when $\ddot{x} = \dot{x} = 0$, we get

$$9e = \frac{f_{dist}}{4}$$

So, the steady state error is given by

$$e = \frac{f_{dist}}{4 \cdot 9} = \frac{4}{36} \approx 0.111$$

2. **For a certain RR manipulator, the equations of motion are given by**

$$\begin{bmatrix} 4 + c_2 & 1 + c_2 \\ 1 + c_2 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -s_2(\dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) \\ s_2\dot{\theta}_1^2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

(a) **Assume that joint 2 is locked at some value θ_2 using brakes and joint 1 is controlled with a PD controller, $\tau_1 = -40\dot{\theta}_1 - 400(\theta_1 - \theta_{1d})$. What is the minimum and maximum inertia perceived at joint 1 as we vary θ_2? What are the corresponding closed-loop frequencies?**

For joint 2 locked ($\ddot{\theta}_2 = \dot{\theta}_2 = 0$), the equation of motion for joint 1 is:

$$(4 + c_2)\ddot{\theta}_1 = \tau_1$$

The inertia seen at joint 1 is the coefficient of the $\ddot{\theta}_1$ term, $(4 + c_2)$. So, this inertia achieves its maximum and minimum values at $\theta_2 = 0$ and $\theta_2 = 180°$:

$$m_{max} = 5, \ m_{min} = 3$$

The closed-loop equation for joint 1 is

$$(4 + c_2)\ddot{\theta}_1 + 40\dot{\theta}_1 + 400(\theta_1 - \theta_{1d}) = 0$$

To get an expression for closed loop frequency, we compare our closed loop equation with the generic system of Equation 7.9 ($m\ddot{x} + b\dot{x} + kx = 0$).

The closed loop frequency is then given by:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{400}{(4 + c_2)}}$$

So, we have

$$\begin{aligned} m = m_{max} \quad &\Rightarrow \quad \omega_{min} = \frac{20}{\sqrt{5}} \\ m = m_{min} \quad &\Rightarrow \quad \omega_{max} = \frac{20}{\sqrt{3}} \end{aligned}$$

(b) **Still assuming that joint 2 is locked, at what values of θ_2 do the minimum and maximum damping ratios occur? What are the minimum and maximum damping ratios?**

To get an expression for damping ratio, we once again compare our closed loop equation with the generic system of Equation 7.9 ($m\ddot{x}+b\dot{x}+kx = 0$). In this case, using Equation 7.12 the closed-loop damping ratio is given by:

$$\xi = \frac{b}{2\sqrt{km}} = \frac{40}{2\sqrt{400m}} = \frac{1}{\sqrt{m}} = \frac{1}{\sqrt{4 + c_2}}$$

So, the minimum and maximum values of ξ occur at $\theta_2 = 0$ and $\theta_2 = 180°$:

$$\begin{aligned} \xi_{min} &= \frac{1}{\sqrt{m_{max}}} = \frac{1}{\sqrt{5}} \\ \xi_{max} &= \frac{1}{\sqrt{m_{min}}} = \frac{1}{\sqrt{3}} \end{aligned}$$

(c) **Now assume that both joints are free to move, and that this system is controlled by a partitioned PD controller, $\tau = \alpha\tau' + \beta$. Design a partitioned, trajectory-following controller (one that tracks a desired position, velocity and acceleration) which will provide a closed-loop frequency of 10 rad/sec**

on joint 1 and 20 rad/sec on joint 2 and be critically damped over the entire workspace. That is, let

$$\tau' = \ddot{\theta}_d - \begin{bmatrix} k'_{v1} & 0 \\ 0 & k'_{v2} \end{bmatrix} (\dot{\theta} - \dot{\theta}_d) - \begin{bmatrix} k'_{p1} & 0 \\ 0 & k'_{p2} \end{bmatrix} (\theta - \theta_d),$$

then find the matrices α and β and the vector τ, along with the necessary gains k'_{v_i} and k'_{p_i}.

The equations of motion are of the form

$$M(\theta)\ddot{\theta} + V(\dot{\theta}, \theta) = \tau$$

to which we apply a vector of torques τ of the form

$$\tau = \alpha\tau' + \beta$$

To make this look like a unit-mass system, we let

$$\alpha = M(\theta),\ \beta = V(\dot{\theta}, \theta)$$

which gives the unit-mass system

$$\ddot{\theta} = \tau'$$

To this system, we apply the control

$$\tau' = \ddot{\theta}_d - \begin{bmatrix} k'_{v1} & 0 \\ 0 & k'_{v2} \end{bmatrix} (\dot{\theta} - \dot{\theta}_d) - \begin{bmatrix} k'_{p1} & 0 \\ 0 & k'_{p2} \end{bmatrix} (\theta - \theta_d),$$

This yields two closed-loop equations

$$\begin{aligned} \ddot{e}_1 + k'_{v_1}\dot{e}_1 + k'_{p_1}e_1 &= 0 \\ \ddot{e}_2 + k'_{v_2}\dot{e}_2 + k'_{p_2}e_2 &= 0 \end{aligned}$$

where e_i is the error at joint i, $e_i = (\theta_i - \theta_{id})$. Now, we need to choose k'_{v_i} and k'_{p_i} to achieve critical damping, and to achieve our desired closed-loop frequencies. For a unit-mass system, we choose

$$\begin{aligned} k'_{p_i} &= \omega_i^2 \\ k'_{v_i} &= 2\xi_i\omega_i \end{aligned}$$

So, we get

$$k'_{p_1} = 100,\ k'_{v_1} = 20$$

$$k'_{p_2} = 400,\ k'_{v_2} = 40$$

(d) **If $\theta_2 = 180°$, what is the steady-state error vector for a given disturbance torque, $\tau_{dist} = [2\ 4]^T$?**

The controlled system, with a disturbance torque τ_{dist} is

$$M(\theta)\ddot{\theta} + V(\dot{\theta}, \theta) = \tau + \tau_{dist}$$

Substituting in our form for $\tau = \alpha\tau' + \beta$ yields

$$M(\theta)\ddot{\theta} - M(\theta)\tau' = \tau_{dist}$$

This has the form

$$M(\theta)\left[\ddot{\mathbf{e}} + K_v'\dot{\mathbf{e}} + K_p'\mathbf{e}\right] = \tau_{dist}$$

where $\mathbf{e}$ is the error vector $\mathbf{e} = \theta - \theta_d$, and K_v' and K_p' are the matrices given by

$$K_v' = \begin{bmatrix} k_{v_1}' & 0 \\ 0 & k_{v_2}' \end{bmatrix}, \; K_p' = \begin{bmatrix} k_{p_1}' & 0 \\ 0 & k_{p_2}' \end{bmatrix}$$

In the steady state ($\ddot{\mathbf{e}} = \dot{\mathbf{e}} = \mathbf{0}$), the equation is

$$M(\theta)K_p'\mathbf{e} = \tau_{dist}$$

which means that the steady state error is

$$\mathbf{e} = (M(\theta)K_p')^{-1}\tau_{dist}$$

For our values, this is:

$$\mathbf{e} = \left(\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 100 & 0 \\ 0 & 400 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 300 & 0 \\ 0 & 400 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{150} \\ \frac{1}{100} \end{bmatrix}$$