Introduction to Robotics (CS223A) Handout

(Winter 2006/2007)

Homework #5 solutions

1. (a) **Derive a formula that transforms an inertia tensor given in some frame $\{C\}$ into a new frame $\{A\}$. The frame $\{A\}$ can differ from frame $\{C\}$ by both translation and rotation. You may assume that frame $\{C\}$ is located at the center of mass.**

Solving this problem involves using the Parallel Axis Theorem to translate the inertia tensor to a frame at a different location, and a similarity transformation to rotate it into the new frame. These operations can be done in either order, as long as we're careful that the vectors we use are expressed in the correct frame. However, it is definitely easier to do the rotation first.

Assume that we have ${}^A_C T$, the transformation from frame $\{C\}$ coordinates to frame $\{A\}$ coordinates, which contains the rotation matrix ${}^A_C R$ and the translation vector ${}^A\mathbf{p}_C$ which locates the origin of frame $\{C\}$ with respect to $\{A\}$. Let's first solve the problem by a rotation followed by a translation. Consider an intermediate frame $\{C'\}$ which has the same origin as $\{C\}$, but whose axes are parallel to frame $\{A\}$. Using a similarity transformation (see p. 134-135 of Lecture Notes), we know that

$$ {}^{C'}I = {}^{C'}_C R \, {}^C I \, {}^{C'}_C R^T $$

However, since frame $\{C'\}$ has the same orientation as frame $\{A\}$, we know that ${}^{C'}_C R = {}^A_C R$, so

$$ {}^{C'}I = {}^A_C R \, {}^C I \, {}^A_C R^T $$

We now have the inertia tensor expressed in the intermediate frame $\{C'\}$. Since $\{C'\}$ is parallel to $\{A\}$, we can use the Parallel Axis Theorem to transform ${}^{C'}I$ to ${}^A I$. To use this theorem, we just need the vector ${}^A\mathbf{p}_{C'}$ that locates the center of frame $\{C'\}$ with respect to $\{A\}$, expressed in frame $\{A\}$, which yields the formula

$$ {}^A I = {}^{C'}I + m\left[({}^A\mathbf{p}_{C'}^T \, {}^A\mathbf{p}_{C'}) I_3 - {}^A\mathbf{p}_{C'} \, {}^A\mathbf{p}_{C'}^T\right] $$

where m is the total mass of the object and I_3 is the 3×3 identity matrix. Since $\{C'\}$ and $\{C\}$ have the same origin, the vector ${}^A\mathbf{p}_{C'}$ is just ${}^A\mathbf{p}_C$. Substituting this value and our previous expression for ${}^{C'}I$ yields:

$$ {}^A I = {}^A_C R \, {}^C I \, {}^A_C R^T + m\left[({}^A\mathbf{p}_C^T \, {}^A\mathbf{p}_C) I_3 - {}^A\mathbf{p}_C \, {}^A\mathbf{p}_C^T\right] $$

Equivalently, we could do this problem with a translation first, and then a rotation. To do that, we can define an intermediate frame $\{A'\}$, which has the same origin as $\{A\}$, but whose axes are parallel to $\{C\}$. We can get the intertia tensor in the intermediate frame by using the Parallel Axis Theorem. To use it, however, we need the vector ${}^{A'}\mathbf{p}_C$ which locates the origin of frame $\{C\}$ with respect to frame $\{A'\}$, **expressed in frame** $\{A'\}$. Using this formula with the vector expressed in frame $\{A\}$ is incorrect. We can get ${}^{A'}\mathbf{p}_C$ by rotating ${}^A\mathbf{p}_C$ with ${}^{A'}_A R = {}^C_A R$, and then simplify:

$$ \begin{aligned} {}^{A'}I &= {}^C I + m\left[({}^{A'}\mathbf{p}_C^T \, {}^{A'}\mathbf{p}_C) I_3 - {}^{A'}\mathbf{p}_C \, {}^{A'}\mathbf{p}_C^T\right] \\ &= {}^C I + m\left[({}^C_A R \, {}^A\mathbf{p}_C)^T ({}^C_A R \, {}^A\mathbf{p}_C) I_3 - ({}^C_A R \, {}^A\mathbf{p}_C)({}^C_A R \, {}^A\mathbf{p}_C)^T\right] \end{aligned} $$

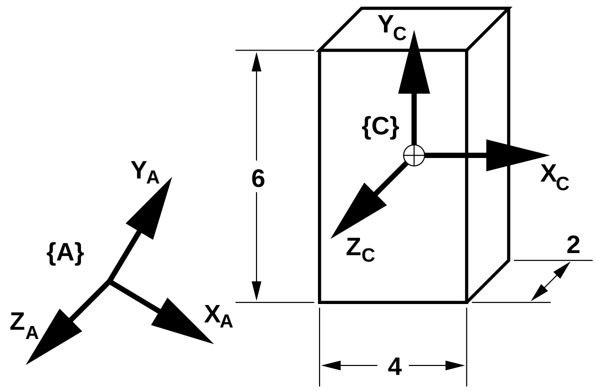
$$
\begin{aligned}
&= {}^C I + m\left[{}^A\mathbf{p}_C^T({}^C_A R^T\, {}^C_A R)\,{}^A\mathbf{p}_C I_3 - {}^C_A R({}^A\mathbf{p}_C\, {}^A\mathbf{p}_C^T)\,{}^C_A R^T\right] \\
&= {}^C I + m\left[{}^A\mathbf{p}_C^T\, {}^A\mathbf{p}_C I_3 - {}^C_A R({}^A\mathbf{p}_C\, {}^A\mathbf{p}_C^T)\,{}^C_A R^T\right]
\end{aligned}
$$

Then, to get the inertia tensor in frame $\{A\}$, we can use a similarity transformation to rotate ${}^{A'}I$:

$$
\begin{aligned}
{}^A I &= {}^A_{A'}R\, {}^{A'}I\, {}^A_{A'}R^T = {}^A_C R\, {}^{A'}I\, {}^A_C R^T \\
&= {}^A_C R\left({}^C I + m\left[{}^A\mathbf{p}_C^T\, {}^A\mathbf{p}_C I_3 - {}^C_A R({}^A\mathbf{p}_C\, {}^A\mathbf{p}_C^T)\,{}^C_A R^T\right]\right){}^A_C R^T \\
&= {}^A_C R\, {}^C I\, {}^A_C R^T + m\, {}^A_C R\left[{}^A\mathbf{p}_C^T\, {}^A\mathbf{p}_C I_3 - {}^C_A R({}^A\mathbf{p}_C\, {}^A\mathbf{p}_C^T)\,{}^C_A R^T\right]{}^A_C R^T \\
&= {}^A_C R\, {}^C I\, {}^A_C R^T + m\left[({}^A\mathbf{p}_C^T\, {}^A\mathbf{p}_C)\,{}^A_C R I_3\, {}^A_C R^T - {}^A_C R\, {}^C_A R({}^A\mathbf{p}_C\, {}^A\mathbf{p}_C^T)\,{}^C_A R^T\, {}^A_C R^T\right] \\
{}^A I &= {}^A_C R\, {}^C I\, {}^A_C R^T + m\left[({}^A\mathbf{p}_C^T\, {}^A\mathbf{p}_C) I_3 - {}^A\mathbf{p}_C\, {}^A\mathbf{p}_C^T\right]
\end{aligned}
$$

This is the same expression that we got from the other approach.

(b) **Consider, for example, the uniform density box shown below. It has mass $m = 12kg$, and dimensions $6m \times 4m \times 2m$:**



Frame $\{C\}$ lies at the center of mass of the box, and the coordinate axes are ligned up with the principal axes of the box. In other words, $\mathbf{Y}_C$ is aligned with the long axis of the box, and $\mathbf{X}_C$ and $\mathbf{Z}_C$ are aligned with the short axes of the box.

Compute the inertia tensor of the box in frame $\{C\}$.

Here, we just put numerical values into the formula given in the homework, to get:

$$
{}^C I = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 52 \end{bmatrix}
$$

(c) **Given the transformation matrix from $\{C\}$ to $\{A\}$:**

$$
{}^A_C T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

use your formula from part (a) and your inertia tensor from part (b) to compute the inertia tensor of the box in frame $\{A\}$.

We apply our formula from part (a). In this case, from ${}^A_CT$, we know:

$$ {}^A_CR = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{p}_C = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} $$

The first part of the transformation (into the intermediate frame $\{A'\}$) is

$$ {}^{A'}I = {}^A_C R^C I^A_C R^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 40 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 52 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 10 & 0 \\ 10 & 30 & 0 \\ 0 & 0 & 52 \end{bmatrix} $$

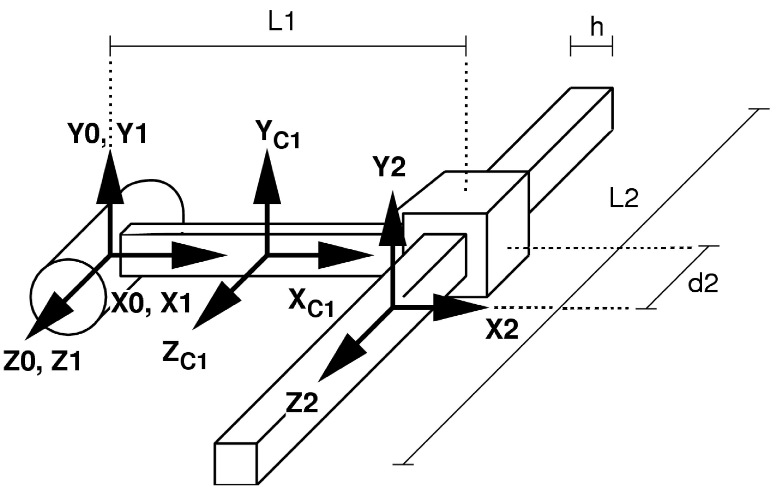
To compute the parallel axis transformation, we need to find the matrix $\left[(\mathbf{p}_C^T\mathbf{p}_C)I_3 - \mathbf{p}_C\mathbf{p}_C^T\right]$:

$$ \mathbf{p}_C^T\mathbf{p}_C = 6, \ \mathbf{p}_C\mathbf{p}_C^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \ \left[(\mathbf{p}_C^T\mathbf{p}_C)I_3 - \mathbf{p}_C\mathbf{p}_C^T\right] = \begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} $$

We now compute the entire transformation:

$$ \begin{aligned} {}^AI &= {}^A_CR^C I^A_C R^T + m\left[(\mathbf{p}_C\mathbf{p}_C^T)I_3 - \mathbf{p}_C\mathbf{p}_C^T\right] \\ &= \begin{bmatrix} 30 & 10 & 0 \\ 10 & 30 & 0 \\ 0 & 0 & 52 \end{bmatrix} + 12 \begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} \\ {}^AI &= \begin{bmatrix} 90 & -2 & -24 \\ -2 & 90 & -24 \\ -24 & -24 & 76 \end{bmatrix} \end{aligned} $$

2. **In the rest of this problem set, we will walk through the process of finding the equations of motion for a simple manipulator from the Lagrange formulation. Consider the RP spatial manipulator shown below. The links of this manipulator are modeled as bars of uniform density, having square cross-sections of thickness h, lengths of L_1 and L_2, and total masses of m_1 and m_2, with centers of mass shown. Assume that the joints themselves are massless.**



From the derivation on pp.131-133 of the notes, we know that the equations of motion have the form:

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] + \mathbf{G}(\mathbf{q}) = \tau$$

where M is the mass matrix, C is the matrix of coefficients for centrifugal forces, B is the matrix of coefficients for Coriolis forces, and G is the vector of gravity forces.

(a) **For each link i, we have attached a frame $\{C_i\}$ to the center of mass (in this case, frame $\{2\}$ is the same as $\{C_2\}$). Compute kinematics for these frames: that is, calculate the matrices ${}^0_{C_1}T$ and ${}^0_{C_2}T$.**

The transformation ${}^1_{C_1}T$ is just a constant offset of $L_1/2$ along the x axis; the other transformations are found in the regular manner:

$${}^0_{C_1}T = \begin{bmatrix} c_1 & -s_1 & 0 & \frac{1}{2}L_1c_1 \\ s_1 & c_1 & 0 & \frac{1}{2}L_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^0_{C_2}T = \begin{bmatrix} c_1 & -s_1 & 0 & L_1c_1 \\ s_1 & c_1 & 0 & L_1s_1 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a two-link manipulator, the mass matrix has the form

$$M = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_1}^T {}^{C_1}I_1 J_{\omega_1} + J_{\omega_2}^T {}^{C_2}I_2 J_{\omega_2}$$

where J_{v_i} is the linear Jacobian of the center of mass of link i, J_{ω_i} is the angular velocity of link i, and ${}^{C_i}I_i$ is the inertia tensor of link i expressed in frame $\{C_i\}$.

(b) **Calculate ${}^0J_{v_1}$ and ${}^0J_{v_2}$.**

These matrices are found directly by differentiating the last columns of ${}^0_{C_i}T$:

$${}^0J_{v_1} = \begin{bmatrix} \frac{\partial {}^0\mathbf{p}_{C_1}}{\partial\theta_1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}L_1s_1 & 0 \\ \frac{1}{2}L_1c_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad {}^0J_{v_2} = \begin{bmatrix} \frac{\partial {}^0\mathbf{p}_{C_2}}{\partial\theta_1} & \frac{\partial {}^0\mathbf{p}_{C_2}}{\partial d_2} \end{bmatrix} = \begin{bmatrix} -L_1s_1 & 0 \\ L_1c_1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) **Calculate ${}^{C_1}J_{\omega_1}$ and ${}^{C_2}J_{\omega_2}$.**

$${}^{C_1}J_{\omega_1} = \begin{bmatrix} \bar{\epsilon}_1 {}^{C_1}\mathbf{z}_1 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad {}^{C_2}J_{\omega_1} = \begin{bmatrix} \bar{\epsilon}_1 {}^{C_2}\mathbf{z}_1 & \bar{\varepsilon}_2 {}^{C_2}\mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

(d) **Calculate ${}^{C_1}I_1$ and ${}^{C_2}I_2$ in terms of the masses and dimensions of the links. You can use the same formula that was given for a box of uniform density in Problem 1(b). Be careful which measurements you use along the axes.**

Using the formula from problem 1, we see that the inertia tensor written at the center of mass of a uniform density rectangular solid is

$${}^CI = \begin{bmatrix} \frac{m}{12}(s_y^2 + s_z^2) & 0 & 0 \\ 0 & \frac{m}{12}(s_x^2 + s_z^2) & 0 \\ 0 & 0 & \frac{m}{12}(s_x^2 + s_y^2) \end{bmatrix}$$

where s_x, s_y and s_z are the dimensions of the solid along the $\mathbf{x}_C$, $\mathbf{y}_C$ and $\mathbf{z}_C$ axes, respectively. Plugging in the values for our links yields

$$^{C_1}I_1 = \begin{bmatrix} \frac{m_1}{6}h^2 & 0 & 0 \\ 0 & \frac{m_1}{12}(L_1^2+h^2) & 0 \\ 0 & 0 & \frac{m_1}{12}(L_1^2+h^2) \end{bmatrix}, \; ^{C_2}I_2 = \begin{bmatrix} \frac{m_2}{12}(L_2^2+h^2) & 0 & 0 \\ 0 & \frac{m_2}{12}(L_2^2+h^2) & 0 \\ 0 & 0 & \frac{m_2}{6}h^2 \end{bmatrix}$$

(e) **Calculate the mass matrix, $M(\mathbf{q})$. To make your algebra easier, leave the inertia tensors in symbolic form until the end, i.e.**

$$^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}$$

This just requires a bit of matrix algebra:

$$^0J_{v_1}^{T\,0}J_{v_1} = \begin{bmatrix} \frac{L_1^2}{4} & 0 \\ 0 & 0 \end{bmatrix}, \; ^0J_{v_2}^{T\,0}J_{v_2} = \begin{bmatrix} L_1^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J_{\omega_1}^{T\,C_1}I_1 J_{\omega_1} = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}, \; J_{\omega_2}^{T\,C_2}I_2 J_{\omega_2} = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} M &= m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_1}^{T\,C_1}I_1 J_{\omega_1} + J_{\omega_2}^{T\,C_2}I_2 J_{\omega_2} \\ &= \begin{bmatrix} \frac{m_1}{4}L_1^2 + m_2(L_1^2) + I_{zz1} + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \\ M &= \begin{bmatrix} \frac{m_1}{3}L_1^2 + \frac{m_1}{12}h^2 + m_2 L_1^2 + \frac{m_2}{6}h^2 & 0 \\ 0 & m_2 \end{bmatrix} \end{aligned}$$

Now we need to calculate the centrifugal and Coriolis forces. We will derive the form directly.

(f) **Beginning with the equation from p. 136 in the lecture notes,**

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_2}\dot{\mathbf{q}} \end{bmatrix},$$

manipulate this equation symbolically into the form

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}]$$

where C and B are matrices in terms of the partial derivatives m_{ijk} of the mass matrix. Don't actually substitute in your answer from part (e) into this equation yet: just leave the elements of these matrices in m_{ijk} symbolic form.

$$\begin{aligned} \mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) &= \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_2}\dot{\mathbf{q}} \end{bmatrix} \\ &= \begin{bmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} [\dot{q}_1 \; \dot{q}_2]\begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \\ [\dot{q}_1 \; \dot{q}_2]\begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} m_{111}\dot q_1 + m_{112}\dot q_2 & m_{121}\dot q_1 + m_{122}\dot q_2 \\ m_{121}\dot q_1 + m_{122}\dot q_2 & m_{221}\dot q_1 + m_{222}\dot q_2 \end{bmatrix} \begin{bmatrix} \dot q_1 \\ \dot q_2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} [\dot q_1\ \dot q_2] \begin{bmatrix} m_{111}\dot q_1 + m_{121}\dot q_2 \\ m_{121}\dot q_1 + m_{221}\dot q_2 \end{bmatrix} \\ [\dot q_1\ \dot q_2] \begin{bmatrix} m_{112}\dot q_1 + m_{122}\dot q_2 \\ m_{122}\dot q_1 + m_{222}\dot q_2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} m_{111}\dot q_1^2 + m_{112}\dot q_1\dot q_2 + m_{121}\dot q_1\dot q_2 + m_{122}\dot q_2^2 \\ m_{121}\dot q_1^2 + m_{122}\dot q_1\dot q_2 + m_{221}\dot q_1\dot q_2 + m_{222}\dot q_2^2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} m_{111}\dot q_1^2 + 2m_{121}\dot q_1\dot q_2 + m_{221}\dot q_2^2 \\ m_{112}\dot q_1^2 + 2m_{122}\dot q_1\dot q_2 + m_{222}\dot q_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}m_{111}\dot q_1^2 + m_{122}\dot q_2^2 - \frac{1}{2}m_{221}\dot q_2^2 + m_{112}\dot q_1\dot q_2 \\ m_{121}\dot q_1^2 - \frac{1}{2}m_{112}\dot q_1^2 + \frac{1}{2}m_{222}\dot q_2^2 + m_{221}\dot q_1\dot q_2 \end{bmatrix}$$

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} \frac{1}{2}m_{111} & m_{122} - \frac{1}{2}m_{221} \\ m_{121} - \frac{1}{2}m_{112} & \frac{1}{2}m_{222} \end{bmatrix} \begin{bmatrix} \dot q_1^2 \\ \dot q_2^2 \end{bmatrix} + \begin{bmatrix} m_{112} \\ m_{221} \end{bmatrix} [\dot q_1 \dot q_2]$$

So we have

$$C = \begin{bmatrix} \frac{1}{2}m_{111} & m_{122} - \frac{1}{2}m_{221} \\ m_{121} - \frac{1}{2}m_{112} & \frac{1}{2}m_{222} \end{bmatrix}, \ B = \begin{bmatrix} m_{112} \\ m_{221} \end{bmatrix}$$

(g) **Using your answer to part (e), compute the matrices $C(\mathbf{q})$ and $B(\mathbf{q})$ in terms of the masses, dimensions, and configuration $\mathbf{q}$ of the manipulator.**

This wasn't meant to be tricky - the mass matrix is independent of the joints, so

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The last thing that remains is to derive the gravity vector $\mathbf{G}(\mathbf{q})$. This you should be able to figure out for yourself.

(h) **Calculate, ${}^0\mathbf{G}(\mathbf{q})$, the gravity vector in frame $\{0\}$, in terms of the masses, the configuration $\mathbf{q}$, and the gravity constant g (g is positive). Assume that gravity pulls things along the $-\mathbf{z}_0$ direction. Be careful with your signs.**

In terms of a unit gravity vector $\mathbf{g}$, we have

$$\mathbf{G} = -\left[J_{v_1}^T m_1 \mathbf{g} + J_{v_2}^T m_2 \mathbf{g}\right]$$

In frame $\{0\}$, the gravity vector is ${}^0\mathbf{g} = [0\ 0\ -g]^T$, which yields

$${}^0\mathbf{G} = -\begin{bmatrix} -\frac{1}{2}L_1 s_1 & \frac{1}{2}L_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -m_1 g \end{bmatrix} - \begin{bmatrix} -L_1 s_1 & L_1 c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -m_2 g \end{bmatrix}$$

$${}^0\mathbf{G} = \begin{bmatrix} 0 \\ m_2 g \end{bmatrix}$$

(i) **As a final step, use your answers to parts (e), (g) and (h) to write out the equations of motion as two great big equations**

$$\tau_1 = f_1(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q})$$
$$\tau_2 = f_2(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q})$$

$$M \begin{bmatrix} \ddot\theta_1 \\ \ddot d_2 \end{bmatrix} + C \begin{bmatrix} \dot\theta_1^2 \\ \dot\theta_2^2 \end{bmatrix} + B\left[\dot\theta_1 \dot d_2\right] + \mathbf{G} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\tau_1 = \left(\frac{m_1}{3}L_1^2 + \frac{m_1}{12}h^2 + m_2 L_1^2 + \frac{m_2}{6}h^2\right)\ddot\theta_1$$

$$\tau_2 = m_2 \ddot d_2 + m_2 g$$