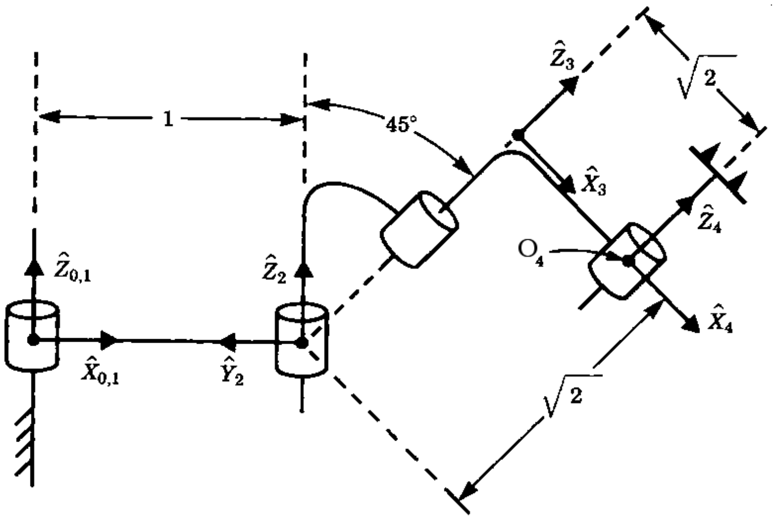
Introduction to Robotics (CS223A) Homework #4 Solution

(Winter 2007/2008)

1. **Consider the following RRRR manipulator (image courtesy J. J. Craig):**



It has the following forward kinematics and rotational Jacobian:

$$
{}^0_4T = \begin{bmatrix} c_{12}c_{34} - \frac{\sqrt{2}}{2}s_{12}s_{34} & -c_{12}s_{34} - \frac{\sqrt{2}}{2}s_{12}c_{34} & \frac{\sqrt{2}}{2}s_{12} & \sqrt{2}c_{12}c_3 - s_{12}(s_3-1) + c_1 \\ s_{12}c_{34} + \frac{\sqrt{2}}{2}c_{12}s_{34} & -s_{12}s_{34} + \frac{\sqrt{2}}{2}c_{12}c_{34} & \frac{\sqrt{2}}{2}c_{12} & \sqrt{2}s_{12}c_3 + c_{12}(s_3-1) + s_1 \\ \frac{\sqrt{2}}{2}s_{34} & \frac{\sqrt{2}}{2}c_{34} & \frac{\sqrt{2}}{2} & s_3 + 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0J_\omega = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 \\ 1 & 1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}
$$

(a) **Find the basic Jacobian J_o in the $\{0\}$ frame, for the position $\mathbf{q} = [0, 90^0, -90^0, 0]^T$. (q is the vector of joint variables.)**

$$
{}^0J_v = \begin{bmatrix} \frac{\partial^0\mathbf{P}_e}{\partial q_1} & \frac{\partial^0\mathbf{P}_e}{\partial q_2} & \frac{\partial^0\mathbf{P}_e}{\partial q_3} & \frac{\partial^0\mathbf{P}_e}{\partial q_4} \end{bmatrix}
$$

where ${}^0\mathbf{P}_e$ is from the 4^{th} column of ${}^0_4T$. Thus:

$$
{}^0J_v = \begin{bmatrix} -\sqrt{2}s_{12}c_3 - c_{12}(s_3-1) - s_1 & -\sqrt{2}s_{12}c_3 - c_{12}(s_3-1) & -\sqrt{2}c_{12}s_3 - s_{12}c_3 & 0 \\ \sqrt{2}c_{12}c_3 - s_{12}(s_3-1) + c_1 & \sqrt{2}c_{12}c_3 - s_{12}(s_3-1) & -\sqrt{2}s_{12}s_3 + c_{12}c_3 & 0 \\ 0 & 0 & c_3 & 0 \end{bmatrix}
$$

Plug in $\mathbf{q} = [0, 90^0, -90^0, 0]^T$, and join with ${}^0J_\omega$ (which was directly given to us) to get:

$$\boxed{{}^0J_o = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 3 & 2 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right]}$$

(b) **A general force vector is applied to the origin of frame $\{4\}$ and measured in frame $\{4\}$ to be $[0, 6, 0, 7, 0, 8]^T$. For the position in (a), determine the joint torques that statically balance it.**

We are given a 6×1 force/moment vector $\mathbf{F}_{app}$ which is exerted on the robot. If the arm is statically balancing this, then we know that the robot must be exerting an equal and opposite force/moment vector at the origin of frame $\{4\}$ (we can thank Sir Isaac Newton for that!).

So we know that in the coordinates of frame $\{4\}$, the vector ${}^4\mathbf{F}_4 = -{}^4\mathbf{F}_{app}$ and we want to find the joint torques τ corresponding to ${}^4\mathbf{F}_4$.

Recall that $\tau = J^T F$. To multiply F and J, however, they must be in the same frame. You can transform either the J from frame $\{0\}$ to $\{4\}$, or transform F from frame $\{4\}$ to $\{0\}$. Both give the same answer.

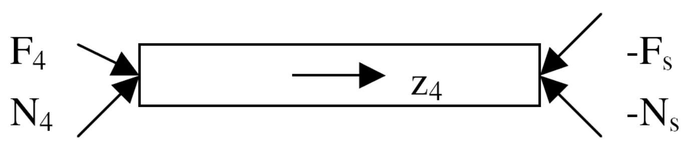
$$\begin{aligned} {}^4\mathbf{F}_4 &= -{}^4\mathbf{F}_{app} = -[0, 6, 0, 7, 0, 8]^T \\ {}^0\mathbf{F}_4 &= \begin{bmatrix} {}^0_4R & 0 \\ 0 & {}^0_4R \end{bmatrix} {}^4\mathbf{F}_4 \\ \tau &= {}^0J^{T\,0}\mathbf{F}_4 \end{aligned}$$

The final answer is:

$$\boxed{\tau = -\,[18.707, 12.707, 16.485, 8.0]^T}$$

(c) **Consider the same configuration as above. A screw driver is gripped in the end-effector so that its tip is along $\hat{Z}_4$ at a distance of 9 units of length from the origin of frame $\{4\}$. What is the force and torque the screw driver tip applies when the same joint torques that were determined in part (b) are applied?**

Let's look at the free-body diagram of the screw-driver, with the left-end being at origin 0_4 and the screw-driver tip on the right. *NOTE: In this diagram, we consider 3x1 force and moment vectors, so "F" represents the 3x1 linear force, NOT the combined 6x1 vector.*



We must first choose an origin for our computations, and then apply static equilibrium. For this computation, *the choice of origin is arbitrary!* You should get the same answer regardless. Two sensible options are either the origin of frame $\{4\}$, or the tip $\{S\}$ of the screw-driver. Let's use the origin of frame $\{4\}$. Also, for simplicity we'll express all our vectors using the coordinates of frame $\{4\}$.

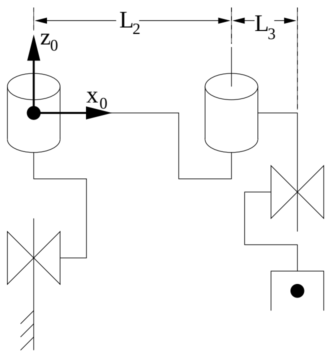
In static equilibrium, we know: $\Sigma\mathbf{F} = 0$ and $\Sigma\mathbf{N} = 0$. These give us:

$$\begin{aligned} \mathbf{F}_4 + (-\mathbf{F}_s) = 0 \quad &\Rightarrow \quad \mathbf{F}_s = \mathbf{F}_4 \\ \mathbf{N}_4 + (-\mathbf{N}_s) + \mathbf{P}_{s4} \times (-\mathbf{F}_s) = 0 \quad &\Rightarrow \quad \mathbf{N}_s = \mathbf{N}_4 +^4 \mathbf{P}_s \times (-\mathbf{F}_s) \end{aligned}$$

The position ${}^4\mathbf{P}_s$ is the position vector from origin $\{4\}$ to the tip, so we know that: ${}^4\mathbf{P}_s = [0, 0, 9]^T$. Meanwhile, from part (b), we that: $\mathbf{F}_4 = -[0, 6, 0]^T$ and $\mathbf{N}_4 = -[7, 0, 8]^T$. If we first solve for $\mathbf{F}_s$ (using the upper equation), we can then use this value to solve for $\mathbf{N}_s$ in the lower equation. THUS:

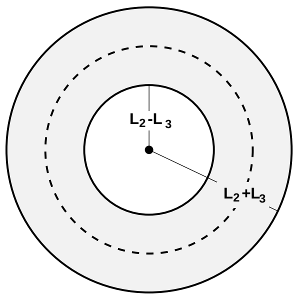
$$\boxed{\begin{aligned} {}^4\mathbf{F}_s &= -[0, 6, 0]^T \\ {}^4\mathbf{N}_s &= -[61, 0, 8]^T \end{aligned}}$$

2. **Consider the PRRP manipulator schematic shown below:**



(a) **Assuming no joint limits, sketch the workspace of this manipulator. Be sure to include dimensions in your drawing. Assume $L_2 > L_3$.**

Since the prismatic joints have no limits, the workspace is an infinite cylinder along the Z_0 direction, whose cross-section is shown in the following figure.



(b) **Describe the (3D) dextrous workspace of this manipulator.**

This manipulator can only point its end-effector downwards, so there are no points for which it can achieve an arbitrary orientation. Even if you consider only the orientation with respect to the (X_0, Y_0) plane (eg. the angle with the X0-axis) there are only two joints to control the position in the plane, leaving no degrees of freedom for controlling the orientation. Therefore, the dexterous workspace is null.

(c) **With no joint limits, if we are considering only the position of the end effector, how many inverse kinematic solutions are there (in general)? Explain briefly.**

If we find a configuration of the joints in this manipulator that places the end-effector at a given position, we can achieve the same position by shortening one prismatic joint and extending the other by any value Δ. This manipulator is redundant in the Z_0 direction, so an infinite number of inverse kinematic solutions exist.

(d) **Imagine that we remove the first prismatic joint, so that the first revolute joint now rotates around the base. Repeat part (c) for such an RRP manipulator.**

If we remove one of the prismatic joints, the manipulator is no longer redundant. For any point (x, y, z), the extension of the prismatic joint is completely determined by z. In the (X_0, Y_0) plane, however, there are two values of the revolute joint angles that will achieve a given (x, y), however: elbow up and elbow down. Therefore, there are two inverse kinematic solutions for a given position.

(e) **Imagine that we further modify the manipulator from part (d) by inserting another revolute joint between the two existing revolute joints, whose axis is oriented in the same direction as the other two. Repeat part (c) for such an RRRP manipulator.**

Compared to part (d), now the manipulator is redundant in the (X_0, Y_0) plane. For a given planar position (x, y), there are three revolute joints for only two position variables (ie. for x and y), thus there are an infinite number of joint angles that will achieve it. This means there are an infinite number of inverse kinematic solutions.

3. **We wish to move a single joint from θ_0 to θ_f, starting and ending at rest, in time t_f. The values of θ_0 and θ_f are given, but we wish to calculate t_f so that these constraints hold: $|\dot{\theta}(t)| < \dot{\theta}_{max}$ and $|\ddot{\theta}(t)| < \ddot{\theta}_{max}$ for all t, where $\dot{\theta}_{max}$ and $\ddot{\theta}_{max}$ are given positive constants.**

(a) **Using a single cubic segment, give equations for the cubic's coefficients a_i in terms of θ_0, θ_f and t_f.**

You can get this right out of the lecture notes or textbook. The long answer goes like this: for the cubic polynomial $\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, we have

$$\begin{aligned} \theta(0) = a_0 &= \theta_0 \\ \theta(t_f) = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 &= \theta_f \\ \dot{\theta}(0) = a_1 &= 0 \\ \dot{\theta}(t_f) = a_1 + 2a_2 t_f + 3a_3 t_f^2 &= 0 \end{aligned}$$

Treating t_f as a constant, the above is just a linear system of four equations and four unknowns (the a_i's), and it can be solved with a little simple algebra to yield:

$$\boxed{\begin{aligned} a_0 &= \theta_0 \\ a_1 &= 0 \\ a_2 &= \tfrac{3(\theta_f - \theta_0)}{t_f^2} \\ a_3 &= \tfrac{-2(\theta_f - \theta_0)}{t_f^3} \end{aligned}}$$

(b) **Using the velocity constraint, $|\dot{\theta}(t)| < \dot{\theta}_{max}$, derive a condition on t_f in terms of θ_0, θ_f, and $\dot{\theta}_{max}$.**

What we can say about $|\dot{\theta}(t)|$ on the interval $[0, t_f]$? First of all, we know that $\dot{\theta}(0) = \dot{\theta}(t_f) = 0$, so $|\dot{\theta}(t)|$ must have its maximum value (in the interval $[0, t_f]$) at some extrema, where $\ddot{\theta}(t) = 0$. This is really just an extreme value problem from your first year calculus class; we find the formula for $\ddot{\theta}(t)$ and set it equal to zero.

The polynomial $\theta(t)$ is given from part (a):

$$\theta(t) = \theta_f + \frac{3(\theta_f - \theta_0)}{t_f^2} t^2 - \frac{2(\theta_f - \theta_0)}{t_f^3} t^3$$

So, taking the first derivative yields

$$\dot{\theta}(t) = \frac{6(\theta_f - \theta_0)}{t_f^2} t - \frac{6(\theta_f - \theta_0)}{t_f^3} t^2$$

And the second derivative is

$$\ddot{\theta}(t) = \frac{6(\theta_f - \theta_0)}{t_f^2} - \frac{12(\theta_f - \theta_0)}{t_f^3} t$$

Setting $\ddot{\theta}(t) = 0$ yields $t = t_f/2$, which makes perfect sense: the velocity is quadratic, and it has equal value at the endpoints of the interval, so its extreme value as at the midpoint of the interval. So, we know that, on the interval $[0, t_f]$,

$$|\dot{\theta}(t)| \leq |\dot{\theta}(t_f/2)|$$

So, in order to make sure that the condition on the maximum velocity is satisfied, we need to make sure that

$$\begin{aligned} |\dot{\theta}\,(t_f/2)| &< \dot{\theta}_{max} \\ \left|\frac{6(\theta_f - \theta_0)}{t_f^2}\,(t_f/2) - \frac{6(\theta_f - \theta_0)}{t_f^3}\,(t_f/2)^2\right| &< \dot{\theta}_{max} \\ \left|\frac{3(\theta_f - \theta_0)}{2t_f}\right| &< \dot{\theta}_{max} \\ \frac{3|\theta_f - \theta_0|}{2t_f} &< \dot{\theta}_{max} \end{aligned}$$

So, our condition is

$$\boxed{t_f > \tfrac{3|\theta_f - \theta_0|}{2\dot{\theta}_{max}}}$$

(c) **Using the acceleration constraint, $|\ddot{\theta}(t)| < \ddot{\theta}_{max}$, derive a condition on t_f in terms of θ_0, θ_f, and $\ddot{\theta}_{max}$.**

This problem is completely analagous to part (b), except that the acceleration is linear, so it will achieve its extreme value at one of the endpoints of the interval. If we plug in $t = 0$ and $t = t_f$ into the acceleration equation, we get

$$\begin{aligned}\ddot{\theta}(0) &= \frac{6(\theta_f - \theta_0)}{t_f^2}\\ \ddot{\theta}(t_f) &= -\frac{6(\theta_f - \theta_0)}{t_f^2}\end{aligned}$$

So, we know that, on the interval $[0, t_f]$,

$$|\ddot{\theta}(t)| \leq |\frac{6(\theta_f - \theta_0)}{t_f^2}| = \frac{6|\theta_f - \theta_0|}{t_f^2}$$

So, in order to make sure that the condition on the maximum acceleration is satisfied, we need to make sure that

$$\frac{6|\theta_f - \theta_0|}{t_f^2} < \ddot{\theta}_{max}$$

So, our condition is

$$\boxed{t_f > \sqrt{\frac{6|\theta_f - \theta_0|}{\ddot{\theta}_{max}}}}$$