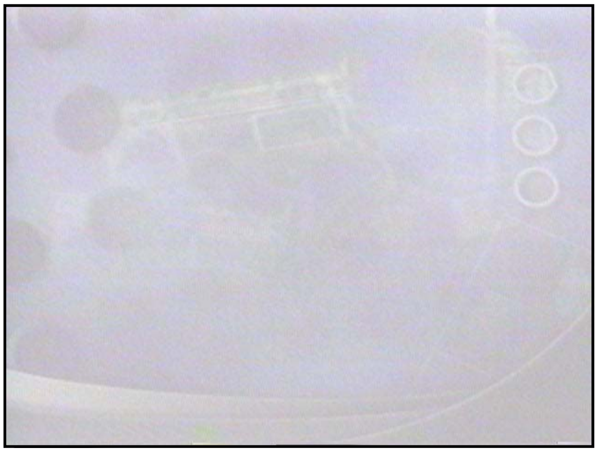
Movie Segment

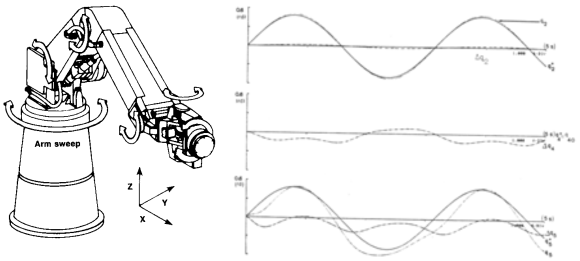
Robotic Reconnaissance Team,
University of Minnesota,
ICRA 2000 video proceedings



Dynamics

- Rigid Body Dynamics
- Newton-Euler Formulation
- Articulated Multi-Body Dynamics
- Recursive Algorithm
- Lagrange Formulation
- Explicit Form

MA23



Joint Space Dynamics

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \Gamma$$

$q:$ Generalized Joint Coordinates

$M(q):$ Mass Matrix - Kinetic Energy Matrix

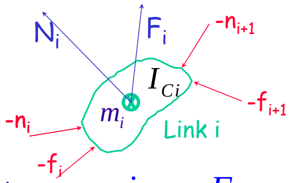
$V(q,\dot{q}):$ Centrifugal and Coriolis forces

$G(q):$ Gravity forces

$\Gamma:$ Generalized forces

Formulations

Newton-Euler



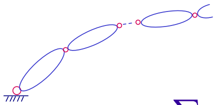
Newton: $m\dot{\mathbf{v}}_C = F$

Euler: $N_i = I_{C_i}\dot{\omega}_i + \omega_i \times I_{C_i}\omega_i$

Eliminate Internal Forces

$$\tau_i = \begin{cases} n_i^T . Z_i & \text{revolute} \\ f_i^T . Z_i & \text{prismatic} \end{cases}$$

Lagrange



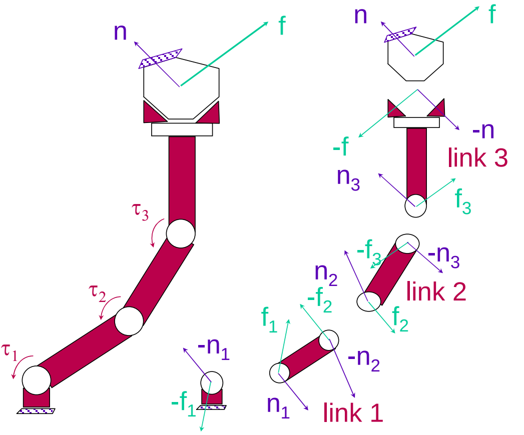
Kinetic Energy: $\sum_i K_i$

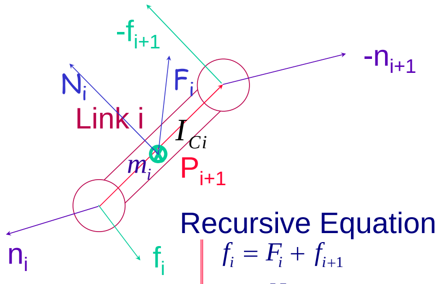
Potential Energy

Generalized Coordinates

$$K = \frac{1}{2}\dot{q}^T M \dot{q}$$

$$M\ddot{q} + V + G = \tau$$





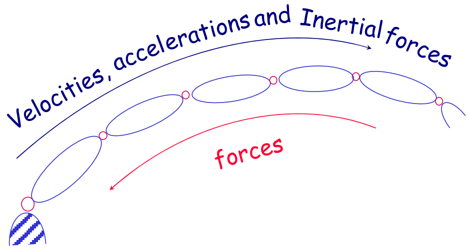
Recursive Equations

$$f_i = F_i + f_{i+1}$$

$$n_i = N_i + n_{i+1} + \mathbf{p}_{C_i} \times F_i + \mathbf{p}_{i+1} \times f_{i+1}$$

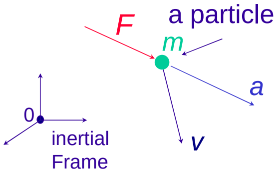
$$\tau_i = \begin{cases} n_i . Z_i & \text{revolute} \\ f_i . Z_i & \text{prismatic} \end{cases}$$

Newton-Euler Algorithm



Newton's Law

$$\underline{F} = m\underline{a}$$



$$\frac{d}{dt}(mv) = F$$

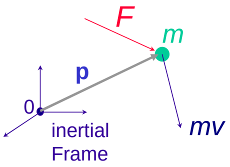
rate of change of the linear momentum is equal to the applied force

Linear Momentum

$$\underline{\varphi = mv}$$

Angular Momentum

$$m\dot{\mathbf{v}} = F$$



take the moment /0

$$\mathbf{p} \times m\dot{\mathbf{v}} = \mathbf{p} \times F$$

N

$$\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = \mathbf{p} \times m\dot{\mathbf{v}} + \overset{=0}{\mathbf{v} \times m\mathbf{v}} = \mathbf{p} \times m\dot{\mathbf{v}}$$

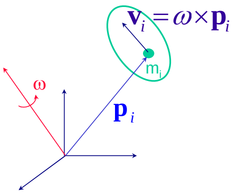
$$\boxed{\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = N}$$

applied moment

angular momentum $\phi = p \times mv$

Rigid Body

Rotational Motion



$\mathbf{v}_i = \omega \times \mathbf{p}_i$

Angular Momentum $= \sum_i \mathbf{p}_i \times m_i \mathbf{v}_i$

$$\phi = \sum_i m_i \mathbf{p}_i \times (\omega \times \mathbf{p}_i)$$

$$m_i \to \rho dv \qquad (\rho\text{: density})$$

$$\phi = \int_V p \times (\omega \times p)\rho dv$$

$$\phi = \int_V p \times (\omega \times p)\rho dv$$

$$\mathbf{p} \times (\omega \times \mathbf{p}) = \hat{\mathbf{p}}(-\hat{\mathbf{p}})\omega$$

$$\phi = [\int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv]\omega$$

Inertia Tensor

$$\underline{\phi = I\omega}$$

Linear Momentum	Angular Momentum
$\underline{\varphi = mv}$	$\underline{\phi = I\omega}$
Newton Equation	Euler Equation
$\frac{d}{dt}(mv) = F$	$\frac{d}{dt}(I\omega) = N$
$\boxed{\dot{\varphi} = F}$	$\boxed{\dot{\phi} = N}$
$ma = F$	$I\dot{\omega} + \omega \times I\omega = N$

Inertia Tensor

$$I = \int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv \qquad (-\hat{\mathbf{p}}\hat{\mathbf{p}}) = (\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T$$

$$I = \int_V [(\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T]\rho dv$$

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad ;\mathbf{p}^T\mathbf{p} = x^2 + y^2 + z^2$$

$$(\mathbf{p}^T\mathbf{p})I_3 = (x^2 + y^2 + z^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{p}\mathbf{p}^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix}(x \quad y \quad z) = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

$$(-\hat{\mathbf{p}}\hat{\mathbf{p}}) = \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & z^2 + x^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$$

Inertia Tensor

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Moments of Inertia →

$$I_{xx} = \iiint (y^2 + z^2)\rho dxdydz$$

$$I_{yy} = \iiint (z^2 + x^2)\rho dxdydz$$

$$I_{zz} = \iiint (x^2 + y^2)\rho dxdydz$$

Products of Inertia →

$$I_{xy} = \iiint xy\rho dxdydz$$

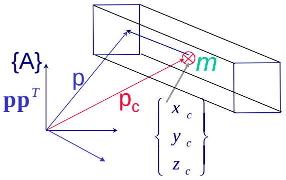
$$I_{xz} = \iiint xz\rho dxdydz$$

$$I_{yz} = \iiint yz\rho dxdydz$$

Parallel Axis theorem

$$I = \int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv$$

$$(-\hat{\mathbf{p}}\hat{\mathbf{p}}) = (\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T$$



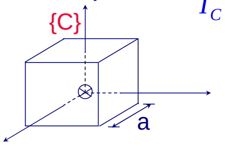
$$I_A = I_C + m\,[(\mathbf{p}_C{}^T\mathbf{p}_C)I_3 - \mathbf{p}_C\mathbf{p}_C{}^T\,]$$

$$I_{A\,zz} = I_{C\,zz} + m(x_C^2 + y_C^2)$$

$$I_{A\,xy} = I_{Cxy} + m x_C y_C$$

Example

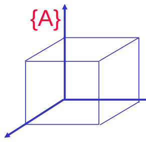
$$I_{C\,zz} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \iint \rho(x^2 + y^2)dxdydz$$



$$I_{Czz} = \frac{1}{6}\rho a^5; \qquad m = \rho a^3$$

$$I_{Cxx} = I_{Cyy} = I_{Czz} = \frac{ma^2}{6}$$

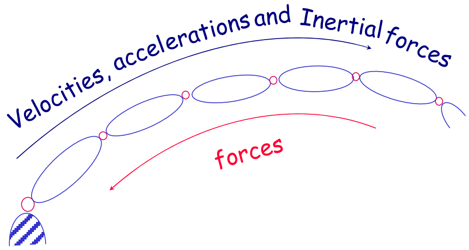
$${}^A x_c = {}^A y_c = {}^A z_c = \frac{a}{2}$$



$$I_{Axx} = I_{Ayy} = I_{Azz} = I_{Czz} + \frac{ma^2}{2} = \frac{2}{3}ma^2$$

$$I_{A\,xy} = I_{A\,xz} = I_{A\,yz} = \frac{m\,a^2}{4}$$

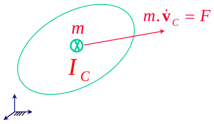
Newton-Euler Algorithm



Newton-Euler Equations

Translational Motion

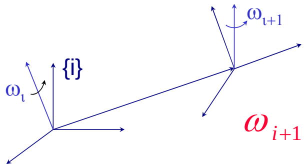
$$m\dot{\mathbf{v}}_C = F$$



Rotational Motion

$$I_C\dot{\omega} + \omega \times I_C\omega = N$$

Angular Acceleration

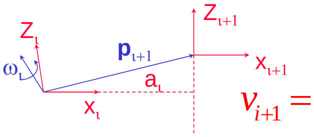


$$\omega_{i+1} = \omega_i + \Omega_{i+1}$$

$$\Omega_{i+1} = \dot{\theta}_{i+1}Z_{i+1}$$

$$\dot{\omega}_{i+1} = \dot{\omega}_i + \dot{\theta}_{i+1}(\omega_i \times Z_{i+1}) + \ddot{\theta}_{i+1}Z_{i+1}$$

Linear Acceleration



$$v_{i+1} = v_i + \omega_i \times p_{i+1} + V_{i+1}$$

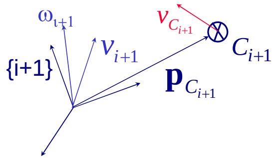
$$V_{i+1} = \dot{d}_{i+1} Z_{i+1}$$

$$P_{i+1} = a_i x_i + d_{i+1} Z_{i+1}$$

$$\dot{\mathbf{v}}_{i+1} = \dot{\mathbf{v}}_i + \dot{\omega}_i \times \mathbf{p}_{i+1} + \omega_i \times \dot{\mathbf{p}}_{i+1} + \dot{V}_{i+1}$$

$$\dot{\mathbf{v}}_{i+1} = \dot{\mathbf{v}}_i + \dot{\omega}_i \times \mathbf{p}_{i+1} + \omega_i \times (\omega_i \times \mathbf{p}_{i+1}) + 2\dot{d}_{i+1}\omega_i \times Z_{i+1} + \ddot{d}_{i+1} Z_{i+1}$$

Velocity and Acceleration at center of mass

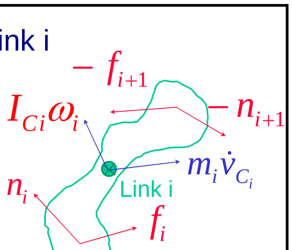


$$\mathbf{v}_{C_{i+1}} = \mathbf{v}_{i+1} + \omega_{i+1} \times \mathbf{p}_{C_{i+1}}$$

$$\dot{\mathbf{v}}_{C_{i+1}} = \dot{\mathbf{v}}_{i+1} + \dot{\omega}_{i+1} \times \mathbf{p}_{C_{i+1}} + \omega_{i+1} \times (\omega_{i+1} \times \mathbf{p}_{C_{i+1}})$$

Dynamic forces on Link i

$$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$$



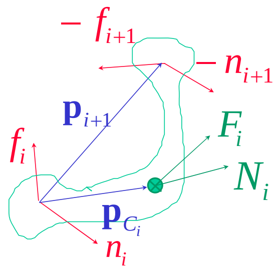
$$m_i \dot{\mathbf{v}}_{C_i} = \sum forces$$

$$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i = \sum moments / c_i$$

Inertial forces/moments

$$F_i = m_i \dot{\mathbf{v}}_{C_i}$$

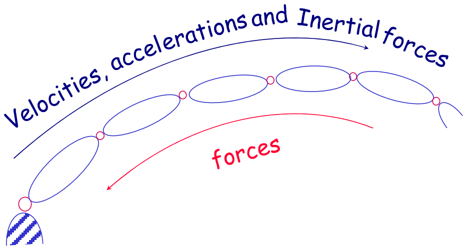
$$N_i = I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$$



$$F_i = f_i - f_{i+1}$$

$$N_i = n_i - n_{i+1} + (-\mathbf{p}_{C_i}) \times f_i + (\mathbf{p}_{i+1} - \mathbf{p}_{C_i}) \times (-f_{i+1}$$

Newton-Euler Algorithm



Recursive Equations

$$f_i = F_i + f_{i+1}$$

$$n_i = N_i + n_{i+1} + \mathbf{p}_{C_i} \times F_i + \mathbf{p}_{i+1} \times f_{i+1}$$

$$\tau_i = \begin{cases} n_i . Z_i & \text{revolute} \\ f_i . Z_i & \text{prismatic} \end{cases}$$

with $F_i = m_i \dot{\mathbf{v}}_{C_i}$

$N_i = I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$

where $\omega_{i+1} = \omega_i + \Omega_{i+1} = \omega_i + \dot{\theta}_{i+1} Z_{i+1}$

$$\dot{\omega}_{i+1} = \dot{\omega}_i + \omega_i \times Z_{i+1}\dot{\theta}_{i+1} + \ddot{\theta}_{i+1} Z_{i+1}$$

$$\dot{v}_{i+1} = \dot{\mathbf{v}}_i + \dot{\omega}_i \times \mathbf{p}_{i+1} + \omega_i \times (\omega_i \times \mathbf{p}_{i+1}) + 2\dot{d}_{i+1}\omega_i \times Z_{i+1} + \ddot{d}_{i+1} Z_{i+1}$$

$$\dot{v}_{C_{i+1}} = \dot{\mathbf{v}}_{i+1} + \dot{\omega}_{i+1} \times \mathbf{p}_{C_{i+1}} + \omega_{i+1} \times (\omega_{i+1} \times \mathbf{p}_{C_{i+1}})$$

Outward iterations: i : 0 ⟶ 5

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}Z_{i+1}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times {}^{i+1}Z_{i+1}\dot{\theta}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}Z_{i+1}$$

$$^{i+1}\dot{\mathbf{v}}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}\mathbf{p}_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}\mathbf{p}_{i+1}) + {}^{i}\dot{\mathbf{v}}_i)$$

$$^{i+1}\dot{\mathbf{v}}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}\mathbf{p}_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}\mathbf{p}_{C_{i+1}}) + {}^{i+1}\dot{\mathbf{v}}_{i+1}$$

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{\mathbf{v}}_{C_{i+1}}$$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}$$

Inward iterations: i : 6 ⟶ 1

$$^{i}f_i = {}_{i+1}^{i}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}N_i + {}_{i+1}^{i}R\,{}^{i+1}n_{i+1} + {}^{i}\mathbf{p}_{C_i} \times {}^{i}F_i + {}^{i}\mathbf{p}_{i+1} \times {}_{i+1}^{i}R\,{}^{i+1}f_{i+1}$$

$$\tau_i = {}^{i}n_i^T\,{}^{i}Z_i$$

Gravity: set $^{0}\dot{\mathbf{v}}_0 = 1G$

Movie Segment

Space Rover, EPFL, Switzerland,
ICRA 2000 video proceedings

Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

Lgrangian

$$L = K - U$$

Kinetic Energy

Potential Energy

Since $U = U(q)$

$$\Rightarrow \underbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q}}_{\text{Inertial forces}} + \frac{\partial U}{\partial q} = \tau$$

Gravity vector

Lagrange Equations

$$\underbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q}}_{\text{Inertial forces}} = \tau - G; \quad G = \frac{\partial U}{\partial q}$$

$$\Downarrow$$

$$M(q)\ddot{q} + V(q,\dot{q}) = \tau - G(q)$$

Inertial forces

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = \tau - G \qquad K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$$\frac{\partial K}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left[\frac{1}{2}\dot{q}^T M(q)\dot{q}\right] = M(q)\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) = \frac{d}{dt}(M\dot{q}) = M\ddot{q} + \dot{M}\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n}\dot{q} \end{bmatrix} = M\ddot{q} + V(q,\dot{q})$$

$$* \frac{\partial K}{\partial \dot{q}} = M\dot{q} \quad \left[K = \frac{1}{2} m\dot{x}^2 ; \frac{\partial}{\partial \dot{x}} (\frac{1}{2} m\dot{x}^2) = 🟩 \right]$$

$$K = \frac{1}{2} \dot{\mathbf{q}}^T M(\mathbf{q}) \dot{\mathbf{q}}$$

$$\mathbf{v} = M^{1/2} \dot{\mathbf{q}} \rightarrow K = \frac{1}{2} \mathbf{v}^T \mathbf{v}$$

$$\frac{\partial K}{\partial \dot{q}} = \frac{\partial K}{\partial v} \frac{\partial v}{\partial \dot{q}} = M^{1/2} v = M\dot{q}$$

$$\frac{\partial}{\partial \mathbf{v}} (\frac{1}{2} \mathbf{v}^T \mathbf{v}) = \mathbf{v} \qquad M^{1/2}$$

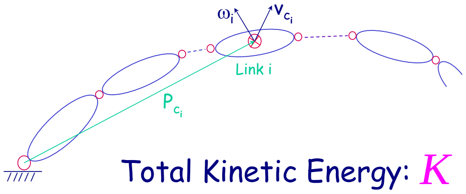
Equations of Motion

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{q}}) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2} \begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1} \dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n} \dot{q} \end{bmatrix} = M\ddot{q} + V(q, \dot{q})$$

$$\boxed{M(q)\ddot{q} + V(q, \dot{q}) + G(q) = \tau}$$

$$M(q): K = \frac{1}{2} \dot{q}^T M \dot{q} \qquad M(q) \quad \Rightarrow \quad V(q, \dot{q})$$

Equations of Motion

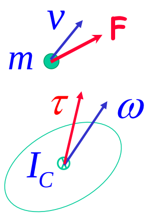


Total Kinetic Energy: K

$$K = \sum K_{Link\,i} \equiv \frac{1}{2} \dot{q}^T M \dot{q}$$

Kinetic Energy

Work done by external forces to bring the system from rest to its current state.

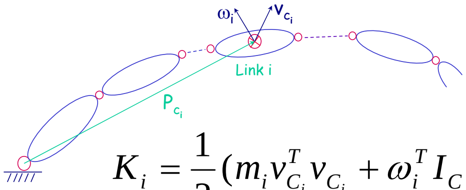


$$K = \frac{1}{2} m v^2$$

$$K = \frac{1}{2} \omega^T I_C \, \omega$$

Equations of Motion

Explicit Form

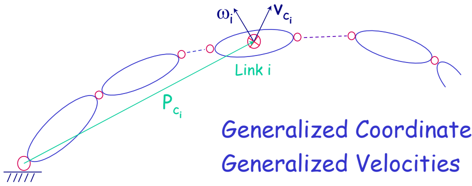


$$K_i = \frac{1}{2} (m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{Ci} \omega_i)$$

Total Kinetic Energy ⇨ $K = \sum_{i=1}^{n} K_i$

Equations of Motion

Explicit Form



Generalized Coordinates q

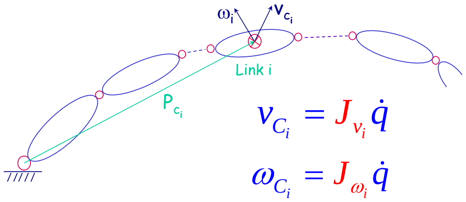
Generalized Velocities $\dot{q}$

Kinetic Energy Quadratic Form of Generalized Velocities $K = \frac{1}{2} \dot{q}^T M \dot{q}$

$$\frac{1}{2} \dot{q}^T M \dot{q} \equiv \frac{1}{2} \sum_{i=1}^{n} (m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{Ci} \omega_i)$$

Equations of Motion

Explicit Form



$$v_{C_i} = J_{v_i}\dot{q}$$

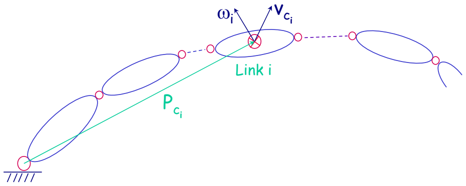
$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

$$\frac{1}{2}\dot{q}^T M\,\dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i}\omega_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n}(m_i\dot{q}^T J_{v_i}^T J_{v_i}\dot{q} + \dot{q}^T J_{\omega_i}^T I_{Ci} J_{\omega_i}\dot{q})$$

Equations of Motion

Explicit Form



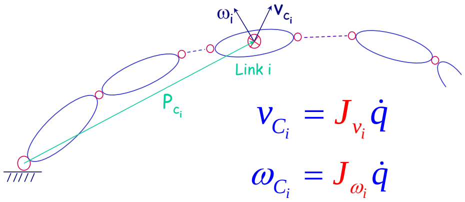
$$\frac{1}{2}\dot{q}^T M\,\dot{q} = \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{Ci} J_{\omega_i})\right]\dot{q}$$

$$\boxed{M = \sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{Ci} J_{\omega_i})}$$

$$\underset{(n\times n)}{M(q)} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & \boxed{m_{nn}} \end{bmatrix}$$

Equations of Motion

Explicit Form



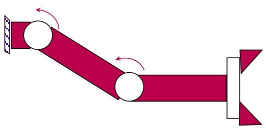
$$v_{C_i} = J_{v_i}\dot{q}$$

$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

$$J_{v_i} = [\frac{\partial p_{C_i}}{\partial q_1} \quad \frac{\partial p_{C_i}}{\partial q_2} \quad \cdots \quad \frac{\partial p_{C_i}}{\partial q_i} \quad 0 \quad 0 \quad \cdots \quad 0]$$

$$J_{\omega_i} = [\bar{\varepsilon}_1 z_1 \quad \bar{\varepsilon}_2 z_2 \quad \cdots \quad \bar{\varepsilon}_i z_i \quad 0 \quad 0 \quad \cdots \quad 0]$$

Vector $V(\mathbf{q},\dot{\mathbf{q}})$ Centrifugal & Coriolis Forces



$$\begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Vector $V(\mathbf{q},\dot{\mathbf{q}})$

$$\frac{\partial M}{\partial q_1} \qquad \frac{\partial m_{11}}{\partial q_1}$$

$$V = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T M_{q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T M_{q_2}\dot{\mathbf{q}} \end{bmatrix} = \begin{pmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{pmatrix}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T\begin{pmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{pmatrix}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T\begin{pmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{pmatrix}\dot{\mathbf{q}} \end{bmatrix}$$

$$\dot{m}_{11} = m_{111}\dot{q}_1 + m_{112}\dot{q}_2$$

$$\frac{\partial m_{22}}{\partial q_2}$$

$$V(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} \frac{1}{2}(m_{111} + m_{111} - m_{111}) & \frac{1}{2}(m_{122} + m_{122} - m_{221}) \\ \frac{1}{2}(m_{211} + m_{211} - m_{112}) & \frac{1}{2}(m_{222} + m_{222} - m_{222}) \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} m_{112} + m_{121} - m_{121} \\ m_{212} + m_{221} - m_{122} \end{bmatrix}[\dot{q}_1\dot{q}_2]$$

Christoffel Symbols

$$\frac{\partial m_{ij}}{\partial q_k}$$

$$b_{ijk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki})$$

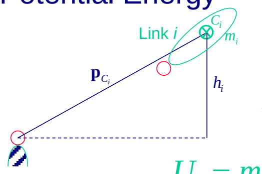
$$V = \begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

$C(\mathbf{q})$ $B(\mathbf{q})$

$$\underset{(n\times n)}{C(\mathbf{q})}\underset{(n\times 1)}{[\dot{\mathbf{q}}^2]} = \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$\underset{(n\times\frac{(n-1)n}{2})}{B(\mathbf{q})} \;\; \underset{(\frac{(n-1)n}{2}\times 1)}{[\dot{\mathbf{q}}\dot{\mathbf{q}}]} = \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \vdots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix} \begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \vdots \\ \dot{q}_{(n-1)}\dot{q}_n \end{bmatrix}$$

Potential Energy



g

$$U_i = m_i g_0 h_i + U_0$$

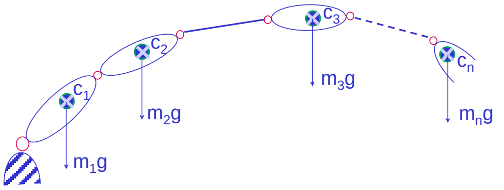
$$U_i = m_i(-g^T \mathbf{p}_{C_i});\; U = \sum_i U_i$$

Gravity Vector

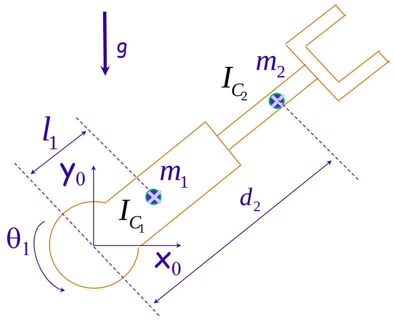
$$G_j = \frac{\partial U}{\partial q_j} = -\sum_{i=1}^{n} (m_i g^T \frac{\partial \mathbf{p}_{C_i}}{\partial q_j})$$

$$G = -\begin{pmatrix} J_{v_1}^T & J_{v_2}^T & \cdots & J_{v_n}^T \end{pmatrix} \begin{pmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{pmatrix}$$

Gravity Vector



$$G = -(J_{v_1}^T(m_1 g) + J_{v_2}^T(m_2 g) + \cdots + J_{v_n}^T(m_n g))$$

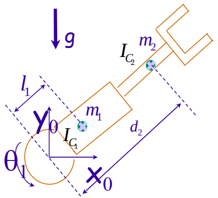


Matrix M

$$M = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T I_{C_1} J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T I_{C_2} J_{\omega_2}$$

J_{v_1} and J_{v_2} : direct differentiation of the vectors:

$${}^0\mathbf{p}_{C_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix};\text{ and } {}^0\mathbf{p}_{C_2} = \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix}$$



In frame {0}, these matrices are:

$${}^0J_{v_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix};\text{ and } {}^0J_{v_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}$$

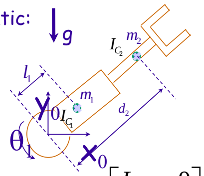
This yields

$$m_1({}^0J_{v_1}^T\,{}^0J_{v_1}) = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix};\text{ and } m_2({}^0J_{v_2}^T\,{}^0J_{v_2}) = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

The matrices J_{ω_1} and J_{ω_2} are given by

$$J_{\omega_1} = [\overline{\in}_1 \mathbf{z}_1 \quad 0] = \text{ and } J_{\omega_2} = [\overline{\in}_1 \mathbf{z}_1 \quad \overline{\in}_2 \mathbf{z}_2]$$

Joint 1 is revolute and joint 2 is prismatic:



$${}^1J_{\omega_1} = {}^1J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

And

$$({}^1J_{\omega_1}^T\,{}^1I_{C_1}\,{}^1J_{\omega_1}) = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix};\text{ and } ({}^1J_{\omega_2}^T\,{}^1I_{C_2}\,{}^1J_{\omega_2}) = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$

Finally,

$$M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$

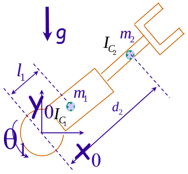
Centrifugal and Coriolis Vector *V*

$$b_{i,jk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right)$$

where $m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$; with $b_{iii} = 0$ and $b_{iji} = 0$ for $i > j$

For this manipulator, only m_{11} is configuration dependent - function of d_2. This implies that only m_{112} is non-zero,

$$m_{112} = 2m_2 d_2 .$$



Matrix *B*
$$B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}.$$

Matrix *C*
$$C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}.$$

Matrix *V*
$$V = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}\left[\dot{\theta}_1 \dot{d}_2\right] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$$

Vector *V*

$$V = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}\left[\dot{\theta}_1 \dot{d}_2\right] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$$

The Gravity Vector *G*

$$\mathbf{G} = -\left[J_{v_1}^T m_1 \mathbf{g} + J_{v_2}^T m_2 \mathbf{g}\right].$$

In frame {0}, ${}^0\mathbf{g} = \begin{pmatrix} 0 & -g & 0 \end{pmatrix}^T$ and the gravity vector is

$${}^0\mathbf{G} = -\begin{bmatrix} -l_1 s_1 & l_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -d_2 s_1 & d_2 c_1 & 0 \\ c_1 & s_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

and

$${}^0\mathbf{G} = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix}$$

Equations of Motion

$$\begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix}$$

$$+\begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}\left[\dot{\theta}_1 \dot{d}_2\right] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}$$

$$+\begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.$$