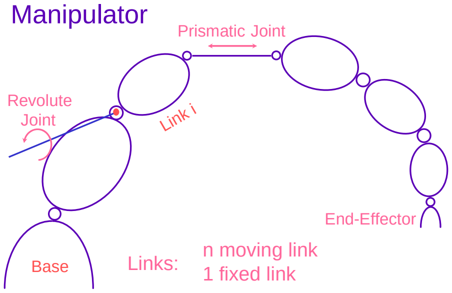
Kinematics

Spatial Descriptions

- Task Description
- Transformations
- Representations

Manipulator

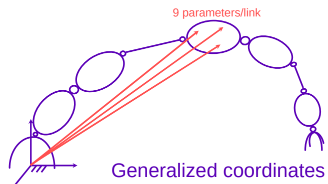


Links: n moving link
1 fixed link

Joints: Revolute (1 DOF)
Prismatic (1 DOF)

Configuration Parameters

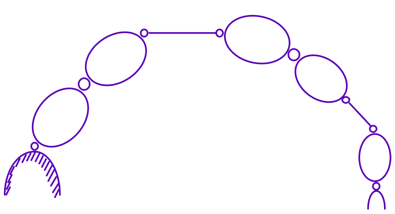
A set of position parameters that describes the full configuration of the system.



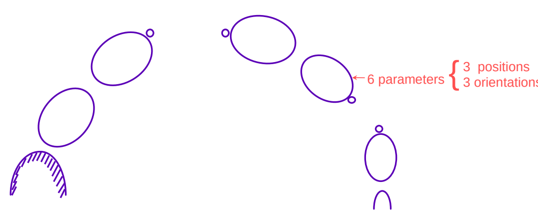
Generalized coordinates
A set of independent configuration parameters

Degrees of Freedom
Number of generalized coordinates

Generalized Coordinates

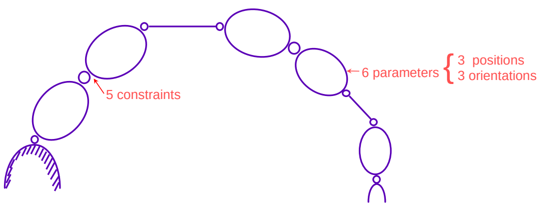


Generalized Coordinates



n moving links: 6n parameters

Generalized Coordinates

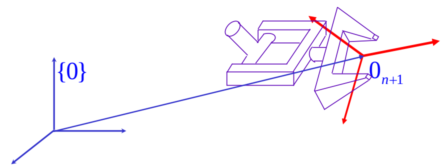


n moving links: 6n parameters

n 1 d.o.f. joints: 5n constraints

$\boxed{\text{d.o.f. (system): } 6n - 5n = n}$

End-Efector Configuration Parameters



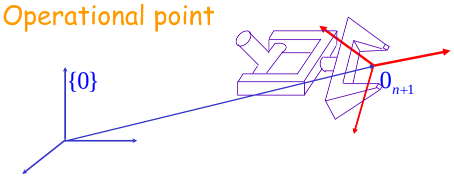
A set of m parameters:

$$(x_1, x_2, x_3, \ldots, x_m)$$

that completely specifies the end-effector position and orientation with respect to $\{0\}$

Operational Coordinates

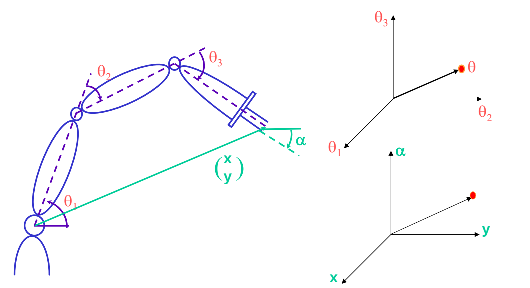
O_{n+1} : Operational point



A set $x_1, x_2, \ldots, x_{m_0}$
of m_0 independent configuration parameters

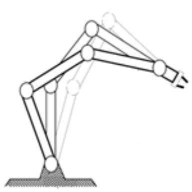
m_0 : number of degrees of freedom of the end-effector.

Joint Coordinates → Joint Space



Operational Coordinates → Operational Space

Redundancy

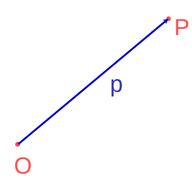


A robot is said to be redundant if

$$\boxed{n > m_0}$$

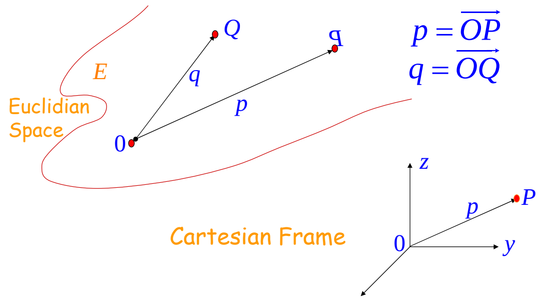
Degrees of redundancy: $n - m_0$

Position of a Point



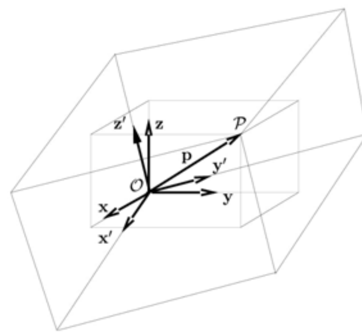
With respect to a fixed origin O, the position of a point P is described by the vector OP or simply by p.

Rigid Body Configuration

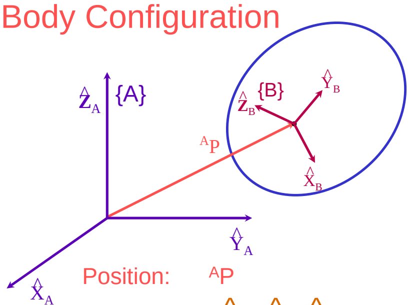


$$p = \overrightarrow{OP}$$
$$q = \overrightarrow{OQ}$$

Coordinate Frames



Rigid Body Configuration



Position: ^{A}P

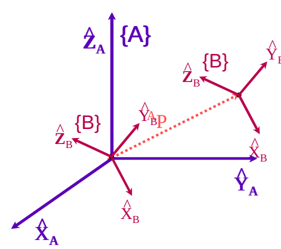
Orientation: $\{^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B\}$

Describe rotations of {B} with respect to {A}

Rotation Matrix

Rotation Matrix

$$^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$



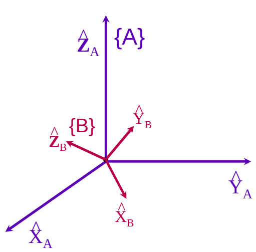
$$^{A}\hat{X}_B = {}^{A}_{B}R\,{}^{B}\hat{X}_B$$

$$^{A}\hat{X}_B = {}^{A}_{B}R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad {}^{A}\hat{Y}_B = {}^{A}_{B}R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad {}^{A}\hat{Z}_B = {}^{A}_{B}R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$^{A}_{B}R = \begin{bmatrix} ^{A}\hat{X}_B & ^{A}\hat{Y}_B & ^{A}\hat{Z}_B \end{bmatrix}$$

Rotation Matrix

$$^{A}_{B}R = \begin{bmatrix} ^{A}\hat{X}_B & ^{A}\hat{Y}_B & ^{A}\hat{Z}_B \end{bmatrix}$$



Dot Product

$$^{A}\hat{X}_B = \begin{bmatrix} \hat{X}_B.\hat{X}_A \\ \hat{X}_B.\hat{Y}_A \\ \hat{X}_B.\hat{Z}_A \end{bmatrix}$$

$$^{A}_{B}R = \begin{bmatrix} \hat{X}_B.\hat{X}_A & \hat{Y}_B.\hat{X}_A & \hat{Z}_B.\hat{X}_A \\ \hat{X}_B.\hat{Y}_A & \hat{Y}_B.\hat{Y}_A & \hat{Z}_B.\hat{Y}_A \\ \hat{X}_B.\hat{Z}_A & \hat{Y}_B.\hat{Z}_A & \hat{Z}_B.\hat{Z}_A \end{bmatrix} \quad {}^{B}X_A^T$$

Rotation Matrix

$$^{A}_{B}R = \begin{bmatrix} ^{A}\hat{X}_B & ^{A}\hat{Y}_B & ^{A}\hat{Z}_B \end{bmatrix} = \begin{bmatrix} ^{B}\hat{X}_A^T \\ ^{B}\hat{Y}_A^T \\ ^{B}\hat{Z}_A^T \end{bmatrix} = \begin{bmatrix} ^{B}\hat{X}_A & ^{B}\hat{Y}_A & ^{B}\hat{Z}_A \end{bmatrix}^T = {}^{B}_{A}R^T$$

$$\underline{\underline{^{A}_{B}R = {}^{B}_{A}R^T}}$$

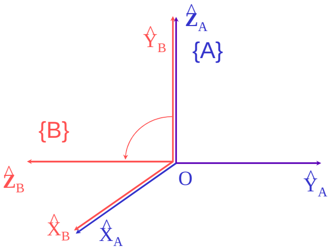
Inverse of Rotation Matrices

$$^{A}_{B}R^{-1} = {}^{B}_{A}R = {}^{A}_{B}R^T$$

$$\boxed{^{A}_{B}R^{-1} = {}^{A}_{B}R^T}$$

Orthonormal Matrix

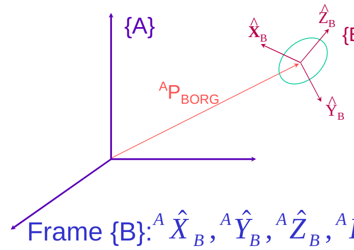
Example



$$ {}^{A}_{B}R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{matrix} \leftarrow {}^{B}\hat{X}_A^T \\ \leftarrow {}^{B}\hat{Y}_A^T \\ \leftarrow {}^{B}\hat{Z}_A^T \end{matrix} $$

$$ \begin{matrix} \uparrow & \uparrow & \uparrow \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{matrix} $$

Description of a Frame



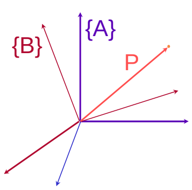
Frame {B}: ${}^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B, {}^{A}P_{Borg}$

$$ \{B\} = \left\{ {}^{A}_{B}R \quad {}^{A}P_{Borg} \right\} $$

Mapping

changing descriptions from frame to frame

Rotations



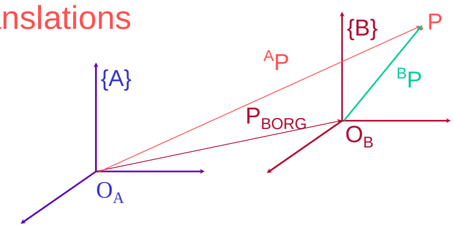
If P is given in {B}: ^{B}P

$$ {}^{A}P = \begin{pmatrix} {}^{B}\hat{X}_A \cdot {}^{B}P \\ {}^{B}\hat{Y}_A \cdot {}^{B}P \\ {}^{B}\hat{Z}_A \cdot {}^{B}P \end{pmatrix} = \begin{pmatrix} {}^{B}\hat{X}_A^T \\ {}^{B}\hat{Y}_A^T \\ {}^{B}\hat{Z}_A^T \end{pmatrix} \cdot {}^{B}P $$

$$ \Downarrow $$

$$ {}^{A}P = {}^{A}_{B}R \; {}^{B}P $$

Translations



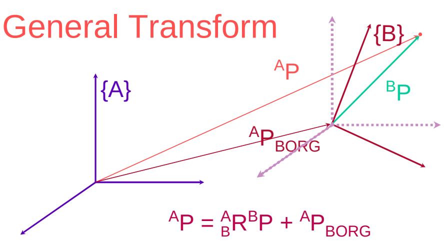
changing the position description of a point P

$\overrightarrow{O_B P} \Rightarrow \overrightarrow{O_A P}$ **(Two different vectors)**

P_{BORG} : $P_{O_B} \Rightarrow P_{O_A}$

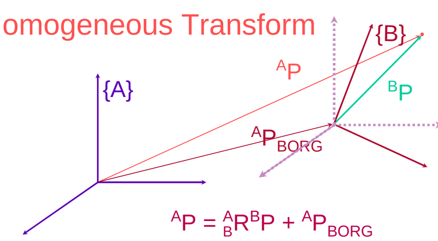
$$ \boxed{ {}^{A}P_{O_A} = {}^{A}P_{O_B} + {}^{A}P_{BORG} } $$

General Transform



$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} $$

Homogeneous Transform

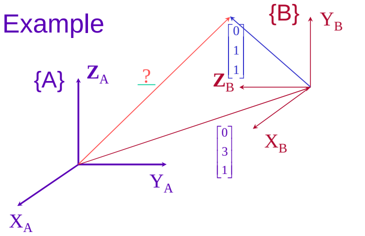


$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} $$

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{Borg} \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array} \right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} $$

$$ \underset{(4x1)}{{}^{A}P} = \underset{(4x4)}{{}^{A}_{B}T} \; \underset{(4x1)}{{}^{B}P} $$

Example



Homogeneous Transform

$$^{A}_{B}T = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & -1 & 3\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix} \quad ^{B}P = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$$

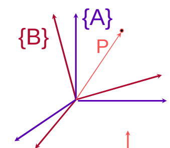
$$^{A}P = {}^{A}_{B}T.{}^{B}P \Rightarrow {}^{A}P = \begin{bmatrix}0\\2\\2\\1\end{bmatrix}$$

Operators

Mapping: changing descriptions from frame to frame

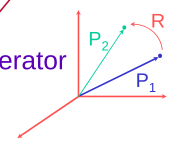
Operators: moving points (within the same frame)

Mapping



$$^{A}P = {}^{A}_{B}R\ {}^{B}P$$

Rotational Operator



$$R: P_1 \longrightarrow P_2$$

$$\boxed{P_2 = R\ P_1}$$

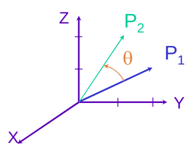
Rotational Operators

$$R_K(\theta): \quad P_1 \longrightarrow P_2$$

$$P_2 = R_K(\theta)\ P_1$$

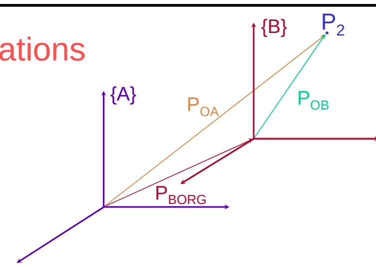
Example

$$R_X(\theta) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$$



$$P_2 = R_X(\theta)P_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 0.8 & -0.6\\ 0 & 0.6 & 0.8\end{bmatrix}\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

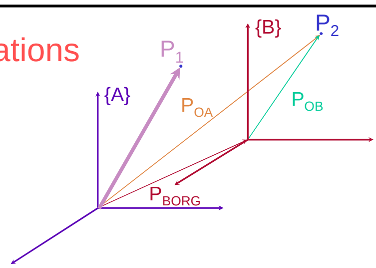
Translations



Mapping: P_{BORG} : $P_{OB} \longrightarrow P_{OA}$ (same point)
2 diff. vectors

$$P_{OA} = P_{OB} + P_{BORG}$$

Translations

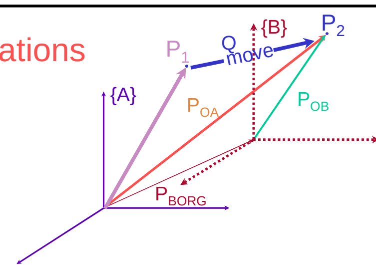


Mapping: P_{BORG} : $P_{OB} \longrightarrow P_{OA}$ (same point)
2 diff. vectors

$$P_{OA} = P_{OB} + P_{BORG}$$

Translational Operator:

Translations



Mapping: P_{BORG} : $P_{OB} \longrightarrow P_{OA}$ (same point)
2 diff. vectors

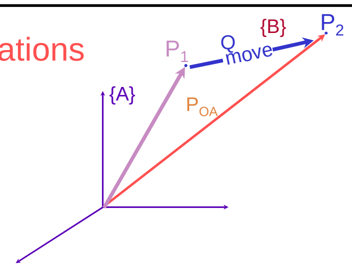
$$P_{OA} = P_{OB} + P_{BORG}$$

Translational Operator:

Q : $P_1 \longrightarrow P_2$ (2 points, 2 diff vectors)

$$P_2 = P_1 + Q$$

Translations

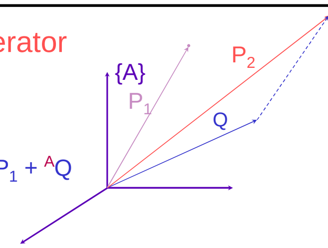


Translational Operator:

Q : $P_1 \longrightarrow P_2$ (2 points, 2 diff vectors)

$P_2 = P_1 + Q$

Translation Operator



Operator: $^{A}P_2 = {}^{A}P_1 + {}^{A}Q$

Homogeneous Transform:

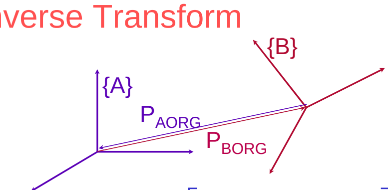
$$D_Q = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^{A}P_2 = {}^{A}D_Q\,{}^{A}P_1$$

General Operators

$$P_2 = \left(\begin{array}{ccc|c} & R_K(\theta) & & Q \\ \hline 0 & 0 & 0 & 1 \end{array}\right) P_1$$

$$\boxed{P_2 = T\,P_1}$$

Inverse Transform



$$^{A}_{B}T = \begin{bmatrix} & ^{A}_{B}R & & ^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R^{-1} = R^{T}$ $(T^{-1} \neq T^{T})$

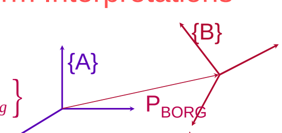
$^{B}P_{AORG}$

$$^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} & ^{A}_{B}R^{T} & & -^{A}_{B}R^{T}.\,^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transform Interpretations

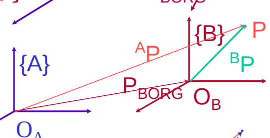
Description of a frame

$^{A}_{B}T$: $\{B\} = \{\,^{A}_{B}R \quad ^{A}P_{Borg}\,\}$



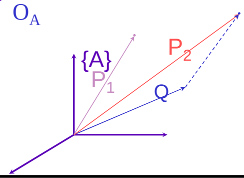
Transform mapping

$^{A}_{B}T$: $^{B}P \rightarrow {}^{A}P$

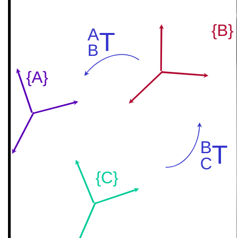


Transform operator

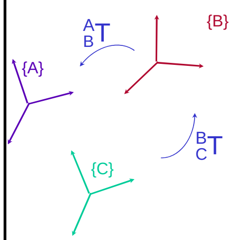
T : $P_1 \rightarrow P_2$



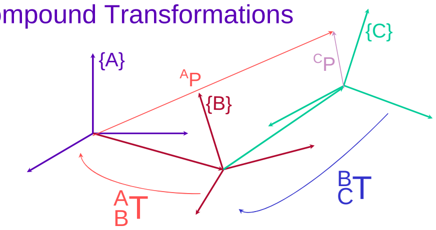
Transform Equation



Transform Equation



Compound Transformations



$$^{B}P = {}^{B}_{C}T\ {}^{C}P$$

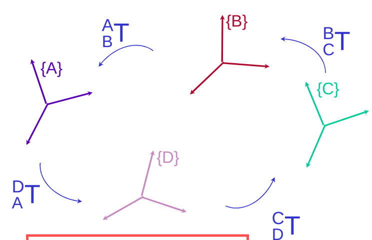
$$^{A}P = {}^{A}_{B}T\ {}^{B}P$$

$$^{A}P = {}^{A}_{B}T\ {}^{B}_{C}T\ {}^{C}P \;\Rightarrow\; {}^{A}_{C}T = {}^{A}_{B}T\ {}^{B}_{C}T$$

$$^{A}_{C}T = {}^{A}_{B}T\ {}^{B}_{C}T$$

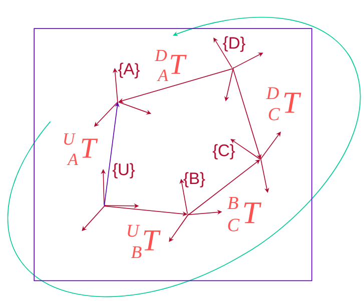
$$^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}_{C}R & {}^{A}_{B}R\,{}^{B}P_{Corg} + {}^{A}P_{Borg} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$$

Transform Equation



$$\boxed{{}^{A}_{B}T\ {}^{B}_{C}T\ {}^{C}_{D}T\ {}^{D}_{A}T \equiv I}$$

$$\Rightarrow\; {}^{B}_{A}T = {}^{B}_{C}T\ {}^{C}_{D}T\ {}^{D}_{A}T$$



$$^{D}_{A}T^{-1}\cdot{}^{D}_{C}T\cdot{}^{B}_{C}T^{-1}\cdot{}^{U}_{B}T^{-1}\cdot{}^{U}_{A}T \equiv I$$

$$^{U}_{A}T = {}^{U}_{B}T\cdot{}^{B}_{C}T\cdot{}^{D}_{C}T^{-1}\cdot{}^{D}_{A}T$$

Spatial Descriptions

- Task Description
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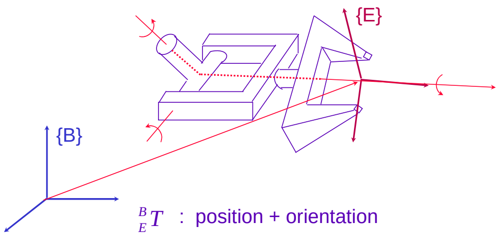
End-Effector Configuration

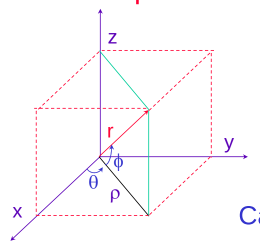


${}^B_E T$: position + orientation

End-Effector Configuration Parameters

$$X = \begin{bmatrix} X_P \\ X_R \end{bmatrix} \begin{matrix} \leftarrow \text{position} \\ \leftarrow \text{orientation} \end{matrix}$$

Position Representations



Cartesian: (x, y, z)

Cylindrical: (ρ, θ, z)

Spherical: (r, θ, ϕ)

Rotation Representations

Rotation Matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix}$$

Direction Cosines

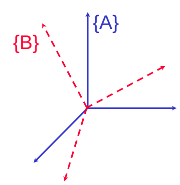
$$x_r = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}_{(9x1)}$$

Constraints

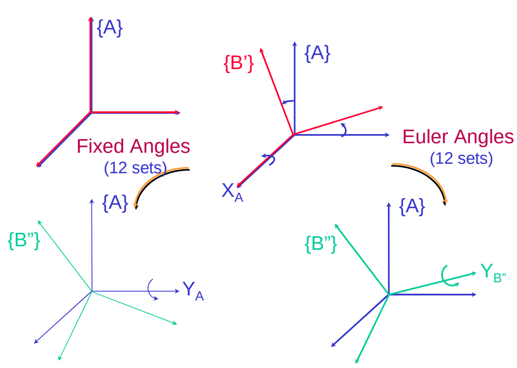
$$|\mathbf{r}_1| = |\mathbf{r}_2| = |\mathbf{r}_3| = 1$$

$$\mathbf{r}_1.\mathbf{r}_2 = \mathbf{r}_1.\mathbf{r}_3 = \mathbf{r}_2.\mathbf{r}_3 = 0$$

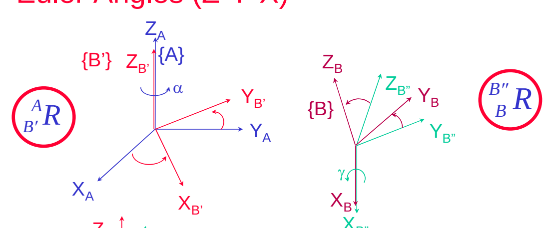
Three Angle Representations

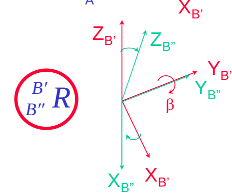


Three Angle Representations



Euler Angles (Z-Y-X)

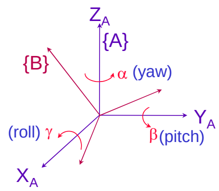




$${}^A_B R = {}^A_{B'}R . {}^{B'}_{B''}R . {}^{B''}_{B}R$$

$$\boxed{{}^A_B R = R_Z(\alpha).R_Y(\beta).R_X(\gamma)}$$

X-Y-Z Fixed Angles



$R_X(\gamma)$: $v \rightarrow R_X(\gamma).v$

$R_Y(\beta)$: $(R_X(\gamma).v) \rightarrow R_Y(\beta).(R_X(\gamma).v)$

$R_Z(\alpha)$: $(R_Y(\beta).R_X(\gamma).v) \rightarrow R_Z(\alpha).(R_Y(\beta).R_X(\gamma).v)$

$$\boxed{{}^A_BR = {}^A_BR_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)}$$

Z-Y-X Euler Angles

$${}^A_BR = R_{Z'}(\alpha).R_{Y'}(\beta).R_{X'}(\gamma)$$

$$\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}.\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

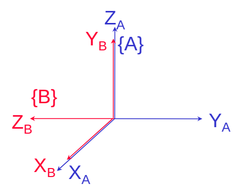
$${}^A_BR = {}^A_BR_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha.c\beta & X & X \\ s\alpha.c\beta & X & X \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix}$$

Z-Y-Z Euler Angles

$${}^A_BR = R_{Z'}(\alpha).R_{Y'}(\beta).R_{Z'}(\gamma)$$

$${}^A_BR = {}^A_BR_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} X & X & c\alpha.s\beta \\ X & X & s\alpha.s\beta \\ -s\beta.c\gamma & s\beta.s\gamma & c\beta \end{bmatrix}$$

Example



$R_{Z'Y'X'}(\alpha,\beta,\gamma)$: $\alpha = 0$, $\beta = 0$, $\gamma = 90^o$

Fixed & Euler Angles

X-Y-Z Fixed Angles

$$R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

Z-Y-X Euler Angles

$$R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

$$\boxed{R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{XYZ}(\gamma,\beta,\alpha)}$$

Inverse Problem

Given ${}^A_BR$ find (α,β,γ)

$$R_{Z'Y'X'}$$

$${}^A_BR = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha.c\beta & c\alpha.s\beta.s\gamma - s\alpha.c\gamma & c\alpha.s\beta.c\gamma + s\alpha.s\gamma \\ s\alpha.c\beta & s\alpha.s\beta.s\gamma + c\alpha.c\gamma & s\alpha.s\beta.c\gamma - c\alpha.s\gamma \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix}$$

$$\left.\begin{aligned} \cos\beta = c\beta = \sqrt{r_{11}^2 + r_{21}^2} \\ \sin\beta = s\beta = -r_{31} \end{aligned}\right\} \rightarrow \beta = A\tan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

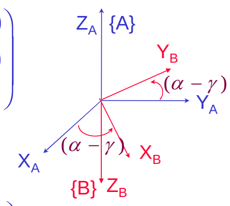
if $c\beta = 0$ $(\beta = \pm 90^o)$ ⇨ Singularity of the representation

⇨ Only $(\alpha + \gamma)$ or $(\alpha - \gamma)$ is defined

Singularities - Example

$c\beta = 0,\ s\beta = +1$

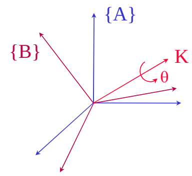
$$ {}^A_BR = \begin{pmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{pmatrix} $$



$c\beta = 0,\ s\beta = -1$

$$ {}^A_BR = \begin{pmatrix} 0 & -s(\alpha+\gamma) & -c(\alpha+\gamma) \\ 0 & c(\alpha+\gamma) & -s(\alpha+\gamma) \\ 1 & 0 & 0 \end{pmatrix} $$

Equivalent angle-axis representation, $R_K(\theta)$



$$ X_r = \theta.K = \begin{bmatrix} \theta.k_X \\ \theta.k_Y \\ \theta.k_Z \end{bmatrix} $$

$$ R_K(\theta) = \begin{bmatrix} k_x.k_x.v\theta + c\theta & k_x.k_y.v\theta - k_z.s\theta & k_x.k_z.v\theta + k_y.s\theta \\ k_x.k_y.v\theta + k_z.s\theta & k_y.k_y.v\theta + c\theta & k_y.k_z.v\theta - k_x.s\theta \\ k_x.k_z.v\theta - k_y.s\theta & k_y.k_z.v\theta + k_x.s\theta & k_z.k_z.v\theta + c\theta \end{bmatrix} $$

with $v\theta = 1 - c\theta$ $\quad R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

$$ \theta = Ar\cos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) $$

$$ {}^AK = \frac{1}{2.\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}, \qquad \text{singularity for } \sin\theta = 0 $$

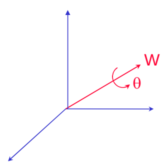
Euler Parameters

$$ \varepsilon_1 = W_x.\sin\frac{\theta}{2} $$

$$ \varepsilon_2 = W_y.\sin\frac{\theta}{2} $$

$$ \varepsilon_3 = W_z.\sin\frac{\theta}{2} $$

$$ \varepsilon_4 = \cos\frac{\theta}{2} $$



Normality Condition

$$ |W| = 1, \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1 $$

ε : point on a unit hypershpere in four-dimensional space

Inverse Problem Given ${}^A_BR$ find ε

$$ \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \equiv \begin{bmatrix} 1 - 2\varepsilon_2^2 - 2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) & 1 - 2\varepsilon_1^2 - 2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) & 1 - 2\varepsilon_1^2 - 2\varepsilon_2^2 \end{bmatrix} $$

$$ r_{11} + r_{22} + r_{33} = 3 - 4(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) $$

$$ (1 - \varepsilon_4^2) $$

$$ \varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} $$

$$ \varepsilon_1 = \frac{r_{32} - r_{23}}{4\varepsilon_4}, \quad \varepsilon_2 = \frac{r_{13} - r_{31}}{4\varepsilon_4}, \quad \varepsilon_3 = \frac{r_{21} - r_{12}}{4\varepsilon_4} $$

$\varepsilon_4 = 0?$

Lemma For all rotations one of the Euler Parameters is greater than or equal to 1/2

$$ \left(\sum_1^4 \varepsilon_i^2 = 1\right) $$

Algorithm Solve with respect to $\max_i\{\varepsilon_i\}$

- $\varepsilon_1 = \max_i\{\varepsilon_i\}$

$$ \varepsilon_1 = \frac{1}{2}\sqrt{r_{11} - r_{22} - r_{33} + 1} $$

$$ \varepsilon_2 = \frac{(r_{21} + r_{12})}{4\varepsilon_1}, \quad \varepsilon_3 = \frac{(r_{31} + r_{13})}{4\varepsilon_1}, \quad \varepsilon_4 = \frac{(r_{32} - r_{23})}{4\varepsilon_1} $$

- $\varepsilon_1 = \max_i\{\varepsilon_i\}$

$$ \varepsilon_1 = \frac{1}{2}\sqrt{r_{11} - r_{22} - r_{33} + 1} $$

- $\varepsilon_2 = \max_i\{\varepsilon_i\}$

$$ \varepsilon_2 = \frac{1}{2}\sqrt{-r_{11} + r_{22} - r_{33} + 1} $$

- $\varepsilon_3 = \max_i\{\varepsilon_i\}$

$$ \varepsilon_3 = \frac{1}{2}\sqrt{-r_{11} - r_{22} + r_{33} + 1} $$

- $\varepsilon_4 = \max_i\{\varepsilon_i\}$

$$ \varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} $$

Euler Parameters / Euler Angles

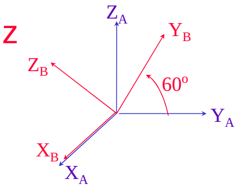
$$\varepsilon_1 = \sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2}$$

$$\varepsilon_2 = \sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}$$

$$\varepsilon_3 = \cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}$$

$$\varepsilon_4 = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}$$

Quiz



Euler Parameters

$$x_r = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ \sqrt{3}/2 \end{bmatrix}$$

Direction Cosines

$$x_r = \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ \sqrt{3}/2 \\ 0 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \end{array} \begin{array}{l} \\ r_1 \\ \\ \\ r_2 \\ \\ \\ r_3 \\ \\ \end{array}$$